



INFINITE SERIES

2.1 Sequences: A sequence of real numbers is defined as a function $f: N \rightarrow R$, where N is a set of natural numbers and R is a set of real numbers. A sequence can be expressed as $\langle f_1, f_2, f_3, \dots, f_n, \dots \rangle$ or $\langle f_n \rangle$. For example $\langle \frac{1}{n} \rangle = \langle \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots \dots \rangle$ is a sequence.

Convergent sequence: A sequence $\langle u_n \rangle$ converges to a number l , if for given $\varepsilon > 0$, there exists a positive integer m depending on ε , such that $|u_n - l| < \varepsilon \forall n \geq m$.

Then l is called the limit of the given sequence and we can write

$$\lim_{n \rightarrow \infty} u_n = l \text{ or } u_n \rightarrow l$$

2.2 Definition of an Infinite Series

An expression of the form $u_1 + u_2 + u_3 + \dots + u_n + \dots$ is known as the infinite series of real numbers, where each u_n is a real number. It is denoted by $\sum_{n=1}^{\infty} u_n$.

For example $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is an infinite series.

Convergence of an infinite series

Consider an infinite series $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \dots$

Let us define $S_1 = u_1$, $S_2 = u_1 + u_2$, $S_3 = u_1 + u_2 + u_3$,,

$$S_n = u_1 + u_2 + u_3 + \dots + u_n \text{ and so on .}$$

Then the sequence $\langle S_n \rangle$ so formed is known as the sequence of partial sums (S.O.P.S.) of the given series.

Convergent series: A series $u_1 + u_2 + u_3 + \dots + u_n + \dots = \sum_{n=1}^{\infty} u_n$ converges if the sequence $\langle S_n \rangle$ of its partial sums converges i.e. if $\lim_{n \rightarrow \infty} S_n$ exists. Also if $\lim_{n \rightarrow \infty} S_n = S$ then S is called as the sum of the given series .

Divergent series: A series $u_1 + u_2 + u_3 + \dots + u_n + \dots = \sum_{n=1}^{\infty} u_n$ diverges if the sequence $\langle S_n \rangle$ of its partial sums diverges i.e. if $\lim_{n \rightarrow \infty} S_n = +\infty$ or $-\infty$.

Example 1 Show that the Geometric series $\sum_{n=1}^{\infty} r^{n-1} = 1 + r + r^2 + r^3 + \dots$, where $r > 0$, is convergent if $r < 1$ and diverges if $r \geq 1$.

Solution: Let us define $S_1 = 1$, $S_2 = 1 + r$, $S_3 = 1 + r + r^2$,,

$$S_n = 1 + r + r^2 + \dots + r^{n-1}$$

Case 1: $r < 1$

$$\begin{aligned} \text{Consider } \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \frac{1-r^n}{1-r} = \frac{1}{1-r} - \lim_{n \rightarrow \infty} \frac{r^n}{1-r} \\ &= \frac{1}{1-r} \quad (\text{As } \lim_{n \rightarrow \infty} r^n = 0 \text{ if } |r| < 1) \end{aligned}$$

Since $\lim_{n \rightarrow \infty} S_n$ is finite $\therefore$ the sequence of partial sums i.e. $\langle S_n \rangle$ converges and hence the given series converges.

Case2: $r > 1$

$$\begin{aligned} \text{Consider } \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \frac{r^n - 1}{r - 1} = \lim_{n \rightarrow \infty} \frac{r^n}{r-1} - \frac{1}{r-1} \\ &\rightarrow \infty \quad (\text{As } r^n \rightarrow \infty \text{ if } r > 1) \end{aligned}$$

Since $\langle S_n \rangle$ diverges and hence the given series diverges.

Case2: $r = 1$

$$\begin{aligned} \text{Consider } S_n &= 1 + r + r^2 + \dots + r^{n-1} \\ &= 1 + 1 + 1 + 1 + \dots + 1 = n \Rightarrow \lim_{n \rightarrow \infty} S_n = \infty \end{aligned}$$

Since $\langle S_n \rangle$ diverges and hence the given series diverges.

Positive term series

An infinite series whose all terms are positive is called a positive term series.

p-series: An infinite series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$ ($p > 0$) is called p-series.

It converges if $p > 1$ and diverges if $p \leq 1$.

For example:

$$1. \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots \text{ converges} \quad (\text{As } p = 3 > 1)$$

$$2. \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} = \frac{1}{1^{5/2}} + \frac{1}{2^{5/2}} + \frac{1}{3^{5/2}} + \dots \text{ converges} \quad (\text{As } p = \frac{5}{2} > 1)$$

$$3. \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \frac{1}{1^{1/2}} + \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} + \dots \text{ diverges} \quad (\text{As } p = \frac{1}{2} < 1)$$

Necessary condition for convergence:

If an infinite series $\sum_{n=1}^{\infty} u_n$ is convergent then $\lim_{n \rightarrow \infty} u_n = 0$. However, converse need not be true.

Proof: Consider the sequence $\langle S_n \rangle$ of partial sums of the series $\sum_{n=1}^{\infty} u_n$.

$$\text{We know that} \quad S_n = u_1 + u_2 + u_3 + \dots + u_n$$

$$= u_1 + u_2 + u_3 + \dots + u_{n-1} + u_n$$

$$\Rightarrow S_{n-1} = u_1 + u_2 + u_3 + \dots + u_{n-1}$$

$$\text{Now} \quad S_n - S_{n-1} = u_n$$

Taking limit $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} (S_n - S_{n-1}) = \lim_{n \rightarrow \infty} u_n$$

$$\Rightarrow \lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} = \lim_{n \rightarrow \infty} u_n \dots\dots\dots(1)$$

As $\sum_{n=1}^{\infty} u_n$ is convergent $\therefore$ sequence $\langle S_n \rangle$ of its partial sums is also convergent.

$$\text{Let } \lim_{n \rightarrow \infty} S_n = l, \text{ then } \lim_{n \rightarrow \infty} S_{n-1} = l$$

Substituting these values in equation (1), we get $\lim_{n \rightarrow \infty} u_n = 0$.

To show that converse may not hold, let us consider the series $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n}$.

$$\text{Here } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

But $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series (As $p = 1$)

Corollary: If $\lim_{n \rightarrow \infty} u_n \neq 0$, then $\sum_{n=1}^{\infty} u_n$ cannot converge.

Example 2 Test the convergence of the series $\sum_{n=1}^{\infty} \cos \frac{1}{n}$

$$\text{Solution: Here } u_n = \cos \frac{1}{n} \Rightarrow \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \cos \frac{1}{n} = 1 \neq 0$$

Hence the given series is not convergent.

Example 3 Test the convergence of the series $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$

$$\text{Solution: Here } u_n = \sqrt{\frac{n}{n+1}} \Rightarrow \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = 1 \neq 0$$

Hence the given series is not convergent.

2.3 Tests for the convergence of infinite series

1. Comparison Test:

Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be two positive term series such that

$$u_n \leq kv_n \quad \forall n \text{ (where } k \text{ is a positive number)}$$

Then (i) If $\sum_{n=1}^{\infty} v_n$ converges then $\sum_{n=1}^{\infty} u_n$ also converges.

(ii) If $\sum_{n=1}^{\infty} u_n$ diverges then $\sum_{n=1}^{\infty} v_n$ also diverges.

Example 4 Test the convergence of the following series

$$(i) \sum_{n=1}^{\infty} \frac{1}{n^n} \quad (ii) \sum_{n=2}^{\infty} \frac{1}{\log n} \quad (iii) \sum_{n=1}^{\infty} \frac{1}{2^{n+x}} \quad \forall x > 0$$

Solution: (i) Here $u_n = \frac{1}{n^n}$ We know that $n^n > 2^n$ for $n > 2$

$$\text{Hence } \frac{1}{n^n} < \frac{1}{2^n} \text{ for } n > 2$$

Now $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series $(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots)$ whose common ratio is $\frac{1}{2}$.

Since $\frac{1}{2} < 1 \therefore \sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent series. Thus by comparison test $\sum_{n=1}^{\infty} \frac{1}{n^n}$ is also convergent.

(ii) Here $u_n = \frac{1}{\log n}$ We know that $\log n < n$ for $n \geq 2$

$$\text{Hence } \frac{1}{\log n} > \frac{1}{n} \text{ for } n \geq 2 \Rightarrow \frac{1}{n} < \frac{1}{\log n} \text{ for } n \geq 2$$

Now $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series (As $p = 1$) . Thus by comparison test $\sum_{n=2}^{\infty} \frac{1}{\log n}$ is also divergent.

(iii) Here $u_n = \frac{1}{2^{n+x}}$. Clearly $2^n + x > 2^n$ (as $x > 0$)

$$\therefore \frac{1}{2^{n+x}} < \frac{1}{2^n}$$

Now $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series $(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots)$ whose common ratio is $\frac{1}{2}$.

Since $\frac{1}{2} < 1 \therefore \sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent series. Thus by comparison test $\sum_{n=1}^{\infty} \frac{1}{2^{n+x}}$ is also convergent.

Example 4 Test the convergence of the series $\sum_{n=1}^{\infty} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2} \right]$

Solution: Here $u_n = \left[\frac{1}{n^2} + \frac{1}{(n+1)^2} \right]$

Clearly $\frac{1}{n^2} + \frac{1}{(n+1)^2} < \frac{1}{n^2} + \frac{1}{n^2} < \frac{1}{n^2}$

Now $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series (As $p = 2 > 1$) . Thus by comparison test $\sum_{n=1}^{\infty} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2} \right]$ is also convergent.

2. Limit Form Test:

Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be two positive term series such that

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = l$ (where l is a finite and non zero number).

Then $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ behave in the same manner i.e. either both converge or both diverge.

Example 5 Test the convergence of the series $\frac{1}{3.7} + \frac{1}{4.9} + \frac{1}{5.11} + \dots$

Solution: Here $u_n = \frac{1}{(n+2)(2n+5)}$

Let $v_n = \frac{1}{n^2}$. Now consider $\frac{u_n}{v_n} = \frac{1}{(n+2)(2n+5)} n^2 = \frac{n^2}{2n^2+9n+10}$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{n^2}{2n^2+9n+10}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2 + \frac{9}{n} + \frac{10}{n^2}} = \frac{1}{2} \text{ (which is a finite and non zero number)}$$

Hence by Limit form test , $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ behave similarly.

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (as $p = 2 > 1$)

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{(n+2)(2n+5)}$ also converges.

Example 6 Test the convergence of the series

$$(i) \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{5}} + \dots (ii) \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$$

Solution: (i) Here $u_n = \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$

Let $v_n = \frac{1}{\sqrt{n}}$. Now consider $\frac{u_n}{v_n} = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+2}}$

$$\begin{aligned} \Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{v_n} &= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+2}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}} \\ &= \frac{1}{\sqrt{2}} \text{ (which is a finite and non zero number)} \end{aligned}$$

Hence by Limit form test , $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ behave similarly.

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (as $p = \frac{1}{2} < 1$)

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$ also diverges.

(ii) Here $u_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

$$= \frac{\sqrt{n+1}-\sqrt{n-1}}{n} \cdot \frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}} = \frac{(n+1)-(n-1)}{n\sqrt{n+1}+\sqrt{n-1}} = \frac{2}{n\sqrt{n+1}+\sqrt{n-1}}$$

Let $v_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$. Now consider $\frac{u_n}{v_n} = \frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}}$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{2\sqrt{n}}{\sqrt{n+1} + \sqrt{n-1}}$$

$$= \lim_{n \rightarrow \infty} \frac{2}{\sqrt{1+\frac{1}{n}} + \sqrt{1-\frac{1}{n}}}$$

= 1 (which is a finite and non zero number)

Hence by Limit form test , $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ behave similarly.

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (as $p = \frac{3}{2} > 1$)

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n-1}}{n}$ also converges.

Example 7 Test the convergence of the series

(i) $\sum_{n=1}^{\infty} \left[(n^3 + 1)^{1/3} - n \right]$ (ii) $\sum_{n=1}^{\infty} \sin \frac{1}{n}$

Solution: (i) Here $u_n = (n^3 + 1)^{1/3} - n = n \left(1 + \frac{1}{n^3} \right)^{1/3} - n$

$$= n \left[1 + \frac{1}{3n^3} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!} \cdot \frac{1}{n^6} + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!} \cdot \frac{1}{n^9} + \dots \right] - n$$

$$= \frac{1}{3n^2} - \frac{1}{9n^5}$$

Let $v_n = \frac{1}{n^2}$.

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{3} \text{ (which is a finite and non zero number)}$$

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (as $p = 2 > 1$)

$\therefore \sum_{n=1}^{\infty} u_n$ also converges (by Limit form test).

(ii) Here $u_n = \sin \frac{1}{n}$. Let $v_n = \frac{1}{n}$.

$$\text{Then } \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}}$$

=1 (which is a finite and non zero number)

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (as $p = 1$)

$\therefore \sum_{n=1}^{\infty} u_n$ also diverges (by Limit form test).

Exercise 2A

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} e^{-n^2}$ Ans. Convergent
2. $\sum_{n=1}^{\infty} \frac{1}{n^2 \log n}$ Ans. Convergent
3. $\sum_{n=1}^{\infty} (\sqrt{n^3 + 1} - \sqrt{n^3})$ Ans. Convergent
4. $\frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \frac{\sqrt{4}}{11} + \dots$ Ans. Divergent
5. $\frac{1}{1.2^2} + \frac{1}{2.3^2} + \frac{1}{3.4^2} + \dots$ Ans. Convergent
6. $\sum_{n=1}^{\infty} ((n^3 + 1)^{1/3} - n)$ Ans. Divergent
7. $\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$ Ans. Divergent
8. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tan \frac{1}{n}$ Ans. Convergent
9. $\frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{3+x} + \dots$ Ans. Divergent
10. $\sum_{n=1}^{\infty} \frac{1}{n-1}$ Ans. Divergent

3. D' Alembert's Ratio Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l < 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l > 1$

(iii) Test fails if $l = 1$

Example 8 Test the convergence of the following series:

$$(i) \frac{1}{3} + \frac{1}{2 \cdot 3^2} + \frac{1}{3 \cdot 3^3} + \frac{1}{4 \cdot 3^4} \dots \dots (ii) \frac{1^2 2^2}{1} + \frac{2^2 3^2}{2!} + \frac{3^2 4^2}{3!} + \frac{4^2 5^2}{4!} \dots \dots (iii) \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Solution: (i) Here $u_n = \frac{1}{n 3^n} \Rightarrow u_{n+1} = \frac{1}{(n+1) 3^{n+1}}$

$$\begin{aligned} \text{Then } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} &= \lim_{n \rightarrow \infty} \frac{n 3^n}{(n+1) 3^{n+1}} = \lim_{n \rightarrow \infty} \frac{n}{(n+1) 3} \\ &= \lim_{n \rightarrow \infty} \frac{1}{3 \left(1 + \frac{1}{n}\right)} = 0 < 1 \end{aligned}$$

Hence by Ratio test ,the given series converges.

(ii) Here $u_n = \frac{n^2(n+1)^2}{n!} \Rightarrow u_{n+1} = \frac{(n+1)^2(n+2)^2}{(n+1)!}$

$$\begin{aligned} \text{Then } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)^2(n+2)^2}{(n+1)!} \cdot \frac{n!}{n^2(n+1)^2} \\ &= \lim_{n \rightarrow \infty} \frac{(n+2)^2}{(n+1)} \cdot \frac{1}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{(n+1)} \cdot \left(\frac{1 + \frac{2}{n}}{1}\right)^2 = 0 < 1 \end{aligned}$$

Hence by Ratio test , the given series converges.

(iii) Here $u_n = \frac{n!}{n^n} \Rightarrow u_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}}$

$$\begin{aligned} \text{Then } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)^{(n+1)}} \cdot \frac{n^n}{n!} = \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} = \frac{1}{2.718} < 1 \end{aligned}$$

Hence by Ratio test , the given series converges.

Example 9 Test the convergence of the following series:

$$(i) \frac{1}{7} + \frac{2!}{7^2} + \frac{3!}{7^3} + \frac{4!}{7^4} \dots \dots (ii) \left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \left(\frac{1.2.3.4}{3.5.7.9}\right)^2 + \dots \dots$$

Solution: (i) Here $u_n = \frac{n!}{7^n} \Rightarrow u_{n+1} = \frac{(n+1)!}{7^{n+1}}$

$$\begin{aligned}\text{Then } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{7^{(n+1)}} \cdot \frac{7^n}{n!} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{7} = \infty > 1\end{aligned}$$

Hence by Ratio test , the given series diverges.

$$(ii) \text{ Here } u_n = \left[\frac{1.2.3.4.....n}{3.5.7.9....(2n+1)} \right]^2 \Rightarrow u_{n+1} = \left[\frac{1.2.3.4.....n(n+1)}{3.5.7.9....(2n+1)(2n+3)} \right]^2$$

$$\text{Then } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{2n+3} \right)^2 = \frac{1}{2^2} = \frac{1}{4} < 1$$

Hence by Ratio test , the given series converges.

Example 10 Test the convergence of the following series:

$$(i) \frac{x}{\sqrt{5}} + \frac{x^3}{\sqrt{7}} + \frac{x^5}{\sqrt{9}} + \frac{x^7}{\sqrt{11}} + \dots \quad (ii) \frac{x}{1.3} + \frac{x^2}{2.4} + \frac{x^3}{3.5} + \frac{x^4}{4.6} + \dots \quad (x > 0)$$

$$\text{Solution: (i) Here } u_n = \frac{x^{2n-1}}{\sqrt{2n+3}} \Rightarrow u_{n+1} = \frac{x^{2n+1}}{\sqrt{2n+5}}$$

$$\text{Then } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{x^{2n+1}}{\sqrt{2n+5}} \frac{\sqrt{2n+3}}{x^{2n-1}} = x^2$$

Hence by Ratio test , the given series converges if $x^2 < 1$ and diverges if $x^2 > 1$.

Test fails if $x^2 = 1$. i.e. $x = 1$

$$\text{When } x = 1, u_n = \frac{1}{\sqrt{2n+3}}$$

$$\text{Let } v_n = \frac{1}{\sqrt{n}}. \text{ Now consider } \frac{u_n}{v_n} = \frac{\sqrt{n}}{\sqrt{2n+3}}$$

$$\begin{aligned}\Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{v_n} &= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{2n+3}} \\ &= \frac{1}{2} \text{ (which is a finite and non zero number)}\end{aligned}$$

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (as $p = \frac{1}{2} < 1$) $\therefore \sum_{n=1}^{\infty} u_n$ also diverges for $x=1$ (by Limit form test).

$\therefore$ the given series converges for $x < 1$ and diverges for $x \geq 1$.

$$(ii) \text{ Here } u_n = \frac{x^n}{n(n+2)} \Rightarrow u_{n+1} = \frac{x^{n+1}}{(n+1)(n+3)}$$

$$\text{Then } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{n(n+2)}{(n+1)(n+3)} x = x$$

Hence by Ratio test, the given series converges if $x < 1$ and diverges if $x > 1$

Test fails if $x = 1$

$$\text{When } x = 1, u_n = \frac{1}{n(n+2)}$$

$$\text{Let } v_n = \frac{1}{n^2}.$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{n^2}{n(n+2)}$$

$$= 1 \text{ (which is a finite and non zero number)}$$

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (as $p = 2 > 1$)

$\therefore \sum_{n=1}^{\infty} u_n$ also converges for $x=1$ (by Limit form test).

$\therefore$ the given series converges for $x \leq 1$ and diverges for $x > 1$.

3. Cauchy's n th Root Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n \rightarrow \infty} (u_n)^{1/n} = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l < 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l > 1$

(iii) Test fails if $l = 1$

Example 11 Test the convergence of the following series:

$$(i) 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} \dots \quad (ii) \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} \quad (iii) \sum_{n=1}^{\infty} 5^{-n-(-1)^n}$$

Solution: (i) Here $u_n = \frac{1}{n^n}$

$$\Rightarrow \lim_{n \rightarrow \infty} (u_n)^{1/n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 < 1$$

Hence by Cauchy's root test, the given series converges.

$$\begin{aligned} (ii) \text{ Here } u_n &= \left(\frac{n}{n+1}\right)^{n^2} \Rightarrow \lim_{n \rightarrow \infty} (u_n)^{1/n} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e} < 1 \end{aligned}$$

Hence by Cauchy's root test, the given series converges.

$$\begin{aligned} (iii) \text{ Here } u_n &= 5^{-n-(-1)^n} \Rightarrow \lim_{n \rightarrow \infty} (u_n)^{1/n} = \lim_{n \rightarrow \infty} 5^{-\{n+(-1)^n\}.1/n} \\ &= \lim_{n \rightarrow \infty} 5^{-\left\{1+\frac{(-1)^n}{n}\right\}} = 5^{-1} \\ &= \frac{1}{5} < 1 \end{aligned}$$

Hence by Cauchy's root test, the given series converges.

Example 12 Test the convergence of the following series:

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \dots$$

Solution: Here $u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$

$$\Rightarrow \lim_{n \rightarrow \infty} (u_n)^{1/n} = \lim_{n \rightarrow \infty} \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1}$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{-1} \left[\left(\frac{n+1}{n} \right)^n - 1 \right]^{-1} \\
&= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^{-1} \left[\left(1 + \frac{1}{n} \right)^n - 1 \right]^{-1} \\
&= \frac{1}{e-1} < 1
\end{aligned}$$

Hence by Cauchy's root test, the given series converges.

Exercise 2B

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{2^n}{n^2+2}$ Ans. Convergent
2. $\sum_{n=1}^{\infty} \frac{n!}{2^{2n-1}}$ Ans. Divergent
3. $\sum_{n=1}^{\infty} \frac{1.2.3 \dots n}{7.10 \dots (3n+4)}$ Ans. Convergent
4. $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!}$ Ans. Convergent
5. $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \dots$ Ans. Convergent if $x < 1$,
divergent if $x \geq 1$
6. $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ Ans. Convergent
7. $\sum_{n=1}^{\infty} \frac{n^{n^2}}{\left(n + \frac{1}{5}\right)^{n^2}}$ Ans. Convergent
8. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ Ans. Convergent
9. $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \dots$ (p > 0) Ans. Convergent
10. $\sum_{n=1}^{\infty} \sqrt{\frac{n-1}{n^3+1}} x^n$ ($x > 0$) Ans. Convergent if $x < 1$,
divergent if $x \geq 1$

4. Raabe's Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l > 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l < 1$

(iii) Test fails if $l = 1$

Example 13 Test the convergence of the following series:

$$(i) \frac{2}{3} + \frac{2.4}{3.5} + \frac{2.4.6}{3.5.7} + \frac{2.4.6.8}{3.5.7.9} + \dots \quad (ii) 1 + \frac{3x}{7} + \frac{3.6x^2}{7.10} + \frac{3.6.9x^3}{7.10.13} + \dots \quad (x > 0)$$

Solution: (i) Here $u_n = \frac{2.4.6 \dots 2n}{1.3.5 \dots (2n+1)} \Rightarrow u_{n+1} = \frac{2.4.6 \dots 2n(2n+2)}{1.3.5 \dots (2n+1)(2n+3)}$

$$\text{Then } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{2n+2}{2n+3} = 1$$

Hence Ratio test fails.

Now applying Raabe's test, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) &= \lim_{n \rightarrow \infty} n \left(\frac{2n+3}{2n+2} - 1 \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{n}{2n+1} \right) = \frac{1}{2} < 1 \end{aligned}$$

Hence by Raabe's test, the given series diverges.

(ii) Ignoring the first term, $u_n = \frac{3.6.9 \dots 3n}{7.10.13 \dots (3n+4)} x^n$

$$\Rightarrow u_{n+1} = \frac{3.6.9 \dots 3n(3n+3)}{7.10.13 \dots (3n+4)(3n+7)} x^{n+1}$$

$$\text{Then } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{3n+3}{3n+7} x = x$$

Hence by Ratio test , the given series converges if $x < 1$ and diverges if $x > 1$

Test fails if $x = 1$

When $x = 1$, $\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$

$$\begin{aligned}\Rightarrow \lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) &= \lim_{n \rightarrow \infty} n \left(\frac{3n+7}{3n+3} - 1 \right) \\ &= \lim_{n \rightarrow \infty} \frac{4n}{3n+3} = \frac{4}{3} > 1\end{aligned}$$

Hence by Raabe's test, the given series converges if $x = 1$

$\therefore$ the given series converges if $x \leq 1$ and diverges if $x > 1$.

4. Logarithmic Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n \rightarrow \infty} n \frac{u_n}{u_{n+1}} = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l > 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l < 1$

Example 14 Test the convergence of the series

$$x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \dots$$

Solution: Here $u_n = \frac{n^n x^n}{n!} \Rightarrow u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$

$$\begin{aligned}\text{Then } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1} x}{(n+1)n^n} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^n x}{n^n} \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n x = e \cdot x\end{aligned}$$

Hence by Ratio test , the given series converges if $ex < 1$ i. e. $x < \frac{1}{e}$
and diverges if $ex > 1$ i. e. $x > \frac{1}{e}$

Test fails if $ex = 1$ i. e. $x = \frac{1}{e}$

Since $\frac{u_{n+1}}{u_n}$ involves $e \therefore$ applying logarithmic test.

$$\frac{u_n}{u_{n+1}} = \frac{1}{\left(1 + \frac{1}{n}\right)^n x}$$

$$\therefore \text{ for } x = \frac{1}{e}, \quad \frac{u_n}{u_{n+1}} = e \cdot \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$$\begin{aligned} \log \left(\frac{u_n}{u_{n+1}} \right) &= \log e - \log \left(1 + \frac{1}{n} \right)^n = 1 - n \log \left(1 + \frac{1}{n} \right) \\ &= 1 - n \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \dots \right) \\ &= \frac{1}{2n} - \frac{1}{3n^2} + \dots \\ &= \lim_{n \rightarrow \infty} n \log \left(\frac{u_n}{u_{n+1}} \right) = \lim_{n \rightarrow \infty} n \left(\frac{1}{2n} - \frac{1}{3n^2} + \dots \right) = \frac{1}{2} < 1 \end{aligned}$$

$\therefore$ By logarithmic test , the series diverges for $x = \frac{1}{e}$.

Hence the given series converges for $x < \frac{1}{e}$ and diverges for $x \geq \frac{1}{e}$.

5. Cauchy's Integral Test

If $u(x)$ is non-negative , integrable and monotonically decreasing function such that $u(n) = u_n$, then if $\int_1^\infty u(x) d(x)$ converges then the series $\sum_{n=1}^\infty u_n$ also converges.

Example 15 Test the convergence of the following series

$$(i) \sum_{n=1}^{\infty} \frac{1}{n^2+1} \quad (ii) \sum_{n=2}^{\infty} \frac{1}{n(\log n)}$$

Solution:(i) Here $u_n = \frac{1}{n^2+1}$.

$$\text{Let } u(x) = \frac{1}{x^2+1}$$

Clearly $u(x)$ is non-negative, integrable and monotonically decreasing function.

$$\begin{aligned} \text{Consider } \int_1^{\infty} \frac{1}{x^2+1} d(x) &= [\tan^{-1}x]_1^{\infty} \\ &= \tan^{-1}\infty - \tan^{-1}1 \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \text{ which is finite.} \end{aligned}$$

Hence $\int_1^{\infty} \frac{1}{x^2+1} d(x)$ converges so $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also converges.

(ii) Here $u_n = \frac{1}{n(\log n)}$.

$$\text{Let } u(x) = \frac{1}{x(\log x)}$$

Clearly $u(x)$ is non-negative, integrable and monotonically decreasing function.

$$\text{Consider } \int_2^{\infty} \frac{1}{x(\log x)} d(x) = \log(\log \infty) - \log(\log 2) = \infty$$

Hence $\sum_{n=2}^{\infty} \frac{1}{n(\log n)}$ diverges.

Exercise 2C

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{2.4.6....(2n+2)}{3.5.7.....(2n+3)} x^{n-1} \quad (x > 0)$

Ans. Convergent if $x < 1$, divergent if $x \geq 1$

2. $\sum_{n=1}^{\infty} \frac{(2n!)}{(n!)^2} x^n \quad (x > 0)$

Ans. Convergent if $x < \frac{1}{4}$, divergent if $x \geq \frac{1}{4}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

Ans. Convergent

4. $\sum_{n=1}^{\infty} \frac{1.3.5....(2n-1)}{2.4.6.....2n} x^n \quad (x > 0)$

Ans. Convergent if $x < 1$, divergent if $x \geq 1$

5. $x^2 + \frac{2^2}{3.4} x^4 + \frac{2^2 4^2}{3.4.5.6} x^6 + \frac{2^2 4^2 6^2}{3.4.5.6.7.8} x^8 + \dots$

Ans. Convergent if $|x| \leq 1$, divergent if $|x| > 1$

6. $1 + \frac{x}{2} + \frac{2!x^2}{3^2} + \frac{3!x^3}{4^3} + \frac{4!x^4}{5^4} + \dots$

Ans. Convergent if $x < e$, divergent if $x \geq e$.

7. $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$

Ans. Convergent

2.4 Alternating Series

An infinite series of the form $u_1 - u_2 + u_3 - u_4 + \dots$ ($u_i > 0 \forall i$)

is called an infinite series.

We write $u_1 - u_2 + u_3 - u_4 + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} u_n$

Leibnitz's Test

The alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ converges if it satisfies the following conditions:

$$(i) u_{n+1} \leq u_n$$

$$(ii) \lim_{n \rightarrow \infty} u_n = 0$$

Example 16 Test the convergence of the following series

$$(i) 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \quad (ii) 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Solution: (i) The given series is $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$. Here $u_n = \frac{1}{n}$

$$\text{Since } \frac{1}{n+1} < \frac{1}{n} \quad \therefore u_{n+1} \leq u_n$$

$$\text{Also } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Hence by Leibnitz's test, the given series converges.

(ii) The given series is $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$. Here $u_n = \frac{1}{n^2}$

$$\text{Since } \frac{1}{(n+1)^2} < \frac{1}{n^2} \quad \therefore u_{n+1} \leq u_n$$

$$\text{Also } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$$

Hence by Leibnitz's test, the given series converges.

Absolute Convergence

A series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |u_n|$ is convergent.

For example $\sum_{n=1}^{\infty} u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots$ is absolutely convergent as $\sum_{n=1}^{\infty} |u_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$ is a convergent series (Since it is a geometric series whose common ratio $\frac{1}{2} < 1$).

Result: Every absolutely convergent series is convergent. But the converse may not be true.

Conditional Convergence

A series which is convergent but not absolutely convergent is called conditionally convergent series.

Example 17 Test the convergence and absolute convergence of the following series:

$$(i) 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \quad (ii) 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$$(iii) \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\log n}$$

Solution: (i) The given series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent by Leibnitz's test.

Now, $\sum_{n=1}^{\infty} |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent
(As $p=1$)

Hence the given series is not absolutely convergent. This is an example of conditionally convergent series.

(ii) The given series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$ is convergent by Leibnitz's test.

Also, $\sum_{n=1}^{\infty} |u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent
(As $p = 2 > 1$)

Hence the given series is absolutely convergent.

(iii) The given series $\sum_{n=2}^{\infty} (-1)^{n+1} u_n$

Here $u_n = \frac{1}{\log n}$. Now $\log x$ is an increasing function $\forall x > 0$

$$\therefore \log(n+2) > \log(n+1)$$

$$\text{or } \frac{1}{\log(n+2)} < \frac{1}{\log(n+1)}$$

$$\therefore u_{n+1} \leq u_n$$

$$\text{Also } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{\log n} = 0$$

Hence by Leibnitz's test, the given series is convergent.

Now for absolute convergence, consider $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{\log n}$

It is a divergent series (as discussed earlier).

Hence the given series is not absolutely convergent. This is an example of conditionally convergent series.

Example 18 Test the convergence of the series:

$$(i) \sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2} \right] \quad (ii) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n2^n}$$

Solution:(i) The given series is $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$

$$\text{Consider } \sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2} \right]$$

$$\text{Now, } \frac{1}{n^2} + \frac{1}{(n+1)^2} < \frac{1}{n^2} + \frac{1}{n^2} = \frac{2}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (As $p = 2 > 1$) $\therefore$ by Comparison test $\sum_{n=1}^{\infty} |u_n|$ is also convergent.

Hence the given series is absolutely convergent and so convergent.

(ii) The given series is $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$

$$\text{Consider } \sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n2^n}$$

$$\text{Here } |u_n| = \frac{1}{n2^n}$$

$$\text{Now } \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n2^n}{(n+1)2^{n+1}} \right| = \frac{1}{2} < 1$$

$\therefore$ by Ratio test $\sum_{n=1}^{\infty} |u_n|$ is convergent or the given series is absolutely convergent and hence convergent.

Example 19 Find the values of x for which the series

$x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$ is absolutely convergent and conditionally convergent.

Solution: The given series is $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = \sum_{n=1}^{\infty} (-1)^{n-1} u_n$

$$\text{Then } |u_n| = \left| \frac{x^{2n-1}}{2n-1} \right|$$

$$\text{Now, } \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}} \right| = \frac{1}{x^2}$$

Thus, by Ratio test $\sum_{n=1}^{\infty} |u_n|$ converges if $x^2 < 1$ i. e. $|x| < 1$, diverges if $x^2 > 1$ i. e. $|x| > 1$ and test fails if $|x| = 1$

When $|x| = 1$ i. e. $x = 1$ or $x = -1$, we have

For $x = 1$,

the given series is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$, which is convergent by Leibnitz's test but not absolutely convergent.

For $x = -1$,

the given series is $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \dots$, which is also convergent by Leibnitz's test but not absolutely convergent.

Hence the given series is absolutely convergent for $|x| < 1$ or

$-1 < x < 1$ and conditionally convergent for $|x| = 1$ i. e.

$x = 1$ or -1 .

Exercise 2D

1. Show that the series $1 - \frac{(2!)^2}{4!} + \frac{(3!)^2}{6!} - \dots$ is convergent.
2. Show that the series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ is absolutely convergent.
3. Test the convergence and absolute convergence of the series
 $1 - \frac{1}{2.3} + \frac{1}{2^2.5} - \frac{1}{2^3.7} \dots$
 Ans. Absolutely convergent
4. Show that the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n+3}$ is conditionally convergent.
5. Test the absolute convergence of the series
 $\sum_{n=1}^{\infty} (-1)^{n-1} (\sqrt{n^2 + 1} - n)$
 Ans. Not absolutely convergent
6. Show that the series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \dots$ converges absolutely.
7. Find the interval of convergence of the series $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} \dots$
 Ans. $0 < x \leq 1$

2.5 EXPANSION OF FUNCTIONS

Taylor Series:

If a function $f(x)$ is infinitely differentiable at the point a then $f(x)$ can be expanded about the point ' a ' as

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \dots$$

Also $f(a+h)$, where h is small, can be expanded as

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \dots$$

Malaurin Series:

It is the special case of Taylor series about the point 0 . Hence the Maclaurin series of $f(x)$ is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

Maclaurin series of standard functions:

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
3. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$
4. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, |x| < 1$
5. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$
6. $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \dots$

Example20 Expand $e^x \cos x$ by Maclaurin series.

Solution : By Maclaurin's expansion , we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots \text{.....} \textcircled{1}$$

$$\text{Here } f(x) = e^x \cos x$$

$$\text{Now } f(0) = e^0 \cos 0 = 1$$

$$f'(x) = e^x \cos x - e^x \sin x$$

$$\Rightarrow f'(0) = e^0 \cos 0 - e^0 \sin 0 = 1$$

$$f''(x) = e^x \cos x - e^x \sin x - e^x \sin x - e^x \cos x$$

$$= -2e^x \sin x$$

$$\Rightarrow f''(0) = -2e^0 \sin 0 = 0$$

$$f'''(x) = -2e^x \cos x - 2e^x \sin x$$

$$\Rightarrow f'''(0) = -2e^0 \cos 0 - 2e^0 \sin 0 = -2$$

Similarly $f^{iv}(0) = -1$ and so on .

Putting these values in ①, we get

$$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{4x^4}{4!} - \dots$$

Example21 Expand $\tan x$ in powers of $(x - \frac{\pi}{4})$ upto first four terms.

Solution: By Taylor's expansion , we have

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots\dots\dots \textcircled{1}$$

Here $f(x) = \tan x$ and $a = \frac{\pi}{4}$

$$\text{Now } f\left(\frac{\pi}{4}\right) = \tan \frac{\pi}{4} = 1$$

$$f'(x) = \sec^2 x$$

$$\Rightarrow f'\left(\frac{\pi}{4}\right) = \sec^2 \frac{\pi}{4} = 2$$

$$f''(x) = 2\sec^2 x \tan x$$

$$\Rightarrow f''\left(\frac{\pi}{4}\right) = 2\sec^2 \left(\frac{\pi}{4}\right) \tan \left(\frac{\pi}{4}\right) = 4$$

$$f'''(x) = 2\sec^4 x + 4 \tan^2 x \sec^2 x$$

$$\Rightarrow f'''\left(\frac{\pi}{4}\right) = 16$$

Putting these values in ①, we get

$$\tan x = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

Example22 Show that $\log \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \dots$

Solution: By Maclaurin's expansion , we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots \dots \dots \textcircled{1}$$

Here $f(x) = \log \sec x$

Now $f(0) = 0$

$$f'(x) = \tan x$$

$$\Rightarrow f'(0) = 0$$

$$f''(x) = \sec^2 x = 1 + \tan^2 x$$

$$\Rightarrow f''(0) = 1$$

$$f'''(x) = 2 \sec x \sec x \tan x$$

$$\Rightarrow f'''(0) = 0$$

Similarly $f^{iv}(0) = 2$ and so on .

Putting these values in $\textcircled{1}$, we get

$$\log \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \dots$$

Example 23 Show that $\sin \left(\frac{\pi}{4} + \theta \right) = \frac{1}{\sqrt{2}} \left(1 + \theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \dots \right)$

Solution: By Taylor's expansion , we have

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \dots \dots \dots \textcircled{1}$$

Here $f(x) = \sin x$, $a = \frac{\pi}{4}$ and $h = \theta$

So $\textcircled{1}$ becomes

$$\sin(a + h) = \sin(a) + h \cos a + \frac{h^2}{2!}(-\sin a) + \frac{h^3}{3!}(-\cos a) + \dots$$

$$\text{or } \sin\left(\frac{\pi}{4} + \theta\right) = \sin\left(\frac{\pi}{4}\right) + \theta \cos \frac{\pi}{4} + \frac{\theta^2}{2!} \left(-\sin \frac{\pi}{4}\right) + \frac{\theta^3}{3!} \left(-\cos \frac{\pi}{4}\right) + \dots$$

$$\text{or } \sin\left(\frac{\pi}{4} + \theta\right) = \frac{1}{\sqrt{2}} \left(1 + \theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \dots\right)$$

Example 24 Estimate the value of $\sqrt{10}$ correct to four places of decimal.

Solution: Let $f(x) = \sqrt{x}$

By Taylor's theorem, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots$$

$$\Rightarrow (a+h)^{1/2} = a^{1/2} + h \left(\frac{d}{dx} x^{1/2}\right)_{at x=a} + \frac{h^2}{2!} \left(\frac{d^2}{dx^2} x^{1/2}\right)_{at x=a} + \frac{h^3}{3!} \left(\frac{d^3}{dx^3} x^{1/2}\right)_{at x=a} + \dots \text{.....} \textcircled{1}$$

Taking $a = 9$ and $h = 1$ in $\textcircled{1}$, we get

$$\begin{aligned} 10^{1/2} &= 9^{1/2} + \left(\frac{1}{2} x^{-1/2}\right)_{at x=9} + \frac{1}{2!} \left(\frac{1(-1)}{2.2} x^{-3/2}\right)_{at x=9} + \\ &\frac{1}{3!} \left(\frac{1(-1)(-3)}{2.2.2} x^{-5/2}\right)_{at x=9} + \dots \\ &= 3 + \frac{1}{2.3} - \frac{1}{8.27} + \dots \\ &= 3.1623(\text{approx.}) \end{aligned}$$

2.6 Approximate Error

Let y be a function of x i.e. $y=f(x)$. If δx is a small change in x then the resulting change in y is denoted by δy and is given by

$$\delta y = \frac{dy}{dx} \delta x \text{ approximately.}$$

Example 25 Find the change in the total surface area of a right circular cone when

(i) the radius is constant but there is a small change in the altitude

(ii) the altitude is constant but there is a small change in the radius.

Solution: Let the radius of the base be r , altitude be h and the change in the altitude be the radius is constant but there is a small change in the altitude δh .

Let S be the total surface area of the cone, then

$$S = \pi r^2 + \pi r \sqrt{r^2 + h^2}$$

(i) If altitude changes then $\delta S = \frac{dS}{dh} \delta h$

$$\text{Now, } \frac{dS}{dh} = 0 + \frac{\pi r}{2} (r^2 + h^2)^{-1/2} \cdot 2h = \frac{\pi r h}{\sqrt{r^2 + h^2}}$$

$$\therefore \delta S = \frac{dS}{dh} \delta h = \frac{\pi r h}{\sqrt{r^2 + h^2}} \delta h \text{ approximately.}$$

(ii) If radius changes then $\delta S = \frac{dS}{dr} \delta r$

$$\text{Now, } \frac{dS}{dr} = 2\pi r + \pi \sqrt{r^2 + h^2} + \frac{2\pi r^2}{2\sqrt{r^2 + h^2}} = 2\pi r + \frac{\pi(2r^2 + h^2)}{\sqrt{r^2 + h^2}}$$

$$\therefore \delta S = \frac{dS}{dr} \delta r = 2\pi r + \frac{\pi(2r^2 + h^2)}{\sqrt{r^2 + h^2}} \delta r \text{ approximately.}$$

Example 26 If a, b, c are the sides of the triangle ABC and S is the semi-perimeter, show that if there is a small error δc in the measurement of side c then the error $\delta \Delta$ in the area Δ of the triangle is given by

$$\delta \Delta = \frac{\Delta}{4} \left(\frac{1}{S} + \frac{1}{S-a} + \frac{1}{S-b} + \frac{1}{S-c} \right) \delta c$$

Solution: We know that $S = \frac{(a+b+c)}{2}$

$$\text{and } \Delta^2 = S(S-a)(S-b)(S-c)$$

$$\text{or } 2 \log \Delta = \log S + \log(S-a) + \log(S-b) + \log(S-c)$$

On differentiating both the sides w.r.t. c , we get

$$\begin{aligned}
\frac{2d\Delta}{\Delta dc} &= \frac{1}{S} \frac{dS}{dc} + \frac{1}{S-a} \frac{d(S-a)}{dc} + \frac{1}{S-b} \frac{d(S-b)}{dc} + \frac{1}{S-c} \frac{d(S-c)}{dc} \\
&= \frac{1}{S} \frac{1}{2} + \frac{1}{2(S-a)} + \frac{1}{2(S-b)} + \frac{1}{(S-c)} \left(\frac{1}{2} - 1 \right) \\
\Rightarrow \frac{d\Delta}{dc} &= \frac{\Delta}{4} \left(\frac{1}{S} + \frac{1}{(S-a)} + \frac{1}{(S-b)} + \frac{1}{(S-c)} \right) \\
\Rightarrow \delta\Delta &= \frac{d\Delta}{dc} \delta c = \frac{\Delta}{4} \left(\frac{1}{S} + \frac{1}{(S-a)} + \frac{1}{(S-b)} + \frac{1}{(S-c)} \right) \delta c
\end{aligned}$$

Example 27 If $T = 2\pi \sqrt{\left(\frac{l}{g}\right)}$ find the error in T corresponding to 2% error in l where g is constant.

Solution: Error in T is given by $\delta T = \frac{dT}{dl} \delta l$

$$\text{Now } \frac{dT}{dl} = \frac{2\pi}{\sqrt{g}} \frac{1}{2\sqrt{l}} \therefore \delta T = \frac{\pi}{\sqrt{g}} \frac{1}{\sqrt{l}} \delta l$$

$$\Rightarrow \frac{\delta T}{T} = \frac{\pi}{\sqrt{g}} \frac{\delta l}{\sqrt{l}} \frac{\sqrt{g}}{2\pi\sqrt{l}} = \frac{1}{2} \frac{\delta l}{l}$$

$$\Rightarrow \frac{\delta T}{T} \cdot 100 = \frac{1}{2} \frac{\delta l}{l} \cdot 100$$

As $\frac{\delta l}{l} \cdot 100 = 2 \therefore \frac{\delta T}{T} \cdot 100 = 1$ Hence error in T is 2%.

Exercise 2E

1. Expand $\tan^{-1}x$ in powers of $(x-1)$.

$$\text{Ans. } \tan^{-1}x = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 + \dots$$

2. Using Taylor's theorem find the approximate value of $f\left(\frac{11}{10}\right)$ where $f(x) = x^3 + 3^2 + 15x - 10$

$$\text{Ans. } 11.461$$

3. Show that $\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots$

4. Show that $\tan\left(\frac{\pi}{4} + x\right) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \dots$ and hence find $\tan 46^\circ$

Ans. 1.0355

5. A soap bubble of radius 2cm shrinks to radius 1.9 cm. Find the decrease in volume and surface area.

Ans. -5.024 cm^3 and -5.024 cm^2

6. If $\log_{10}4 = 0.6021$, find the approximate value of $\log_{10}404$.

Ans. 2.61205

7. Let A, B and C be the angles of a triangle opposite to the sides a, b and c respectively. If small errors δa , δb and δc are made in the sides then show that $\delta A = \frac{a}{2\Delta}(\delta a - \delta b \cos C - \delta c \cos B)$ where Δ is the area of the triangle.

