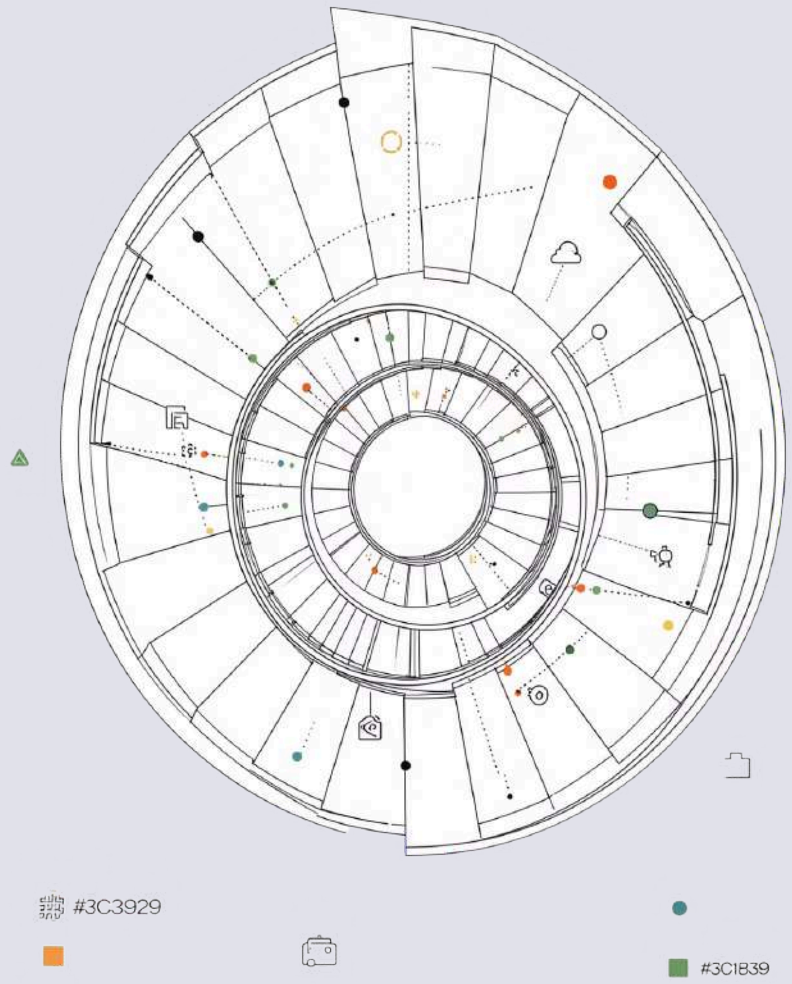
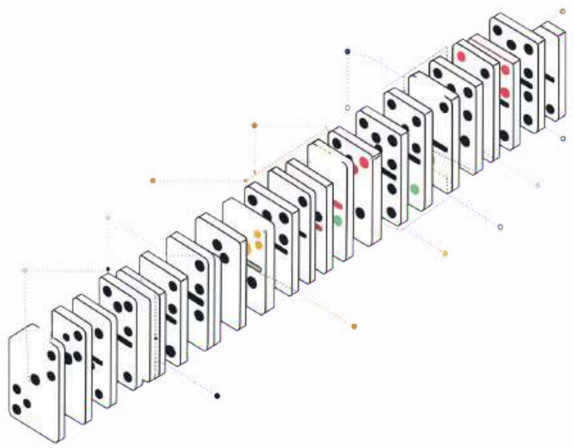
Infinite Series and Convergence

A Calculus II guide to geometric series, p-series, and convergence tests.



What Is an Infinite Series?



An infinite series is a sum of infinitely many terms:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

We study it through its **partial sums**:

$$S_N = \sum_{n=1}^{N} a_n$$

A series **converges** if $\lim_{N\to\infty} S_N$ exists and is finite. Otherwise, it **diverges**.

The Divergence Test

The Test

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ **diverges**.

Key Insight

Terms **must** approach 0 for convergence — but this is *necessary, not sufficient*. Terms going to 0 does not guarantee convergence.

Example

$\sum \frac{n}{n+1}$ diverges because:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

Geometric Series

One of the most important and recognizable series in calculus:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

Converges

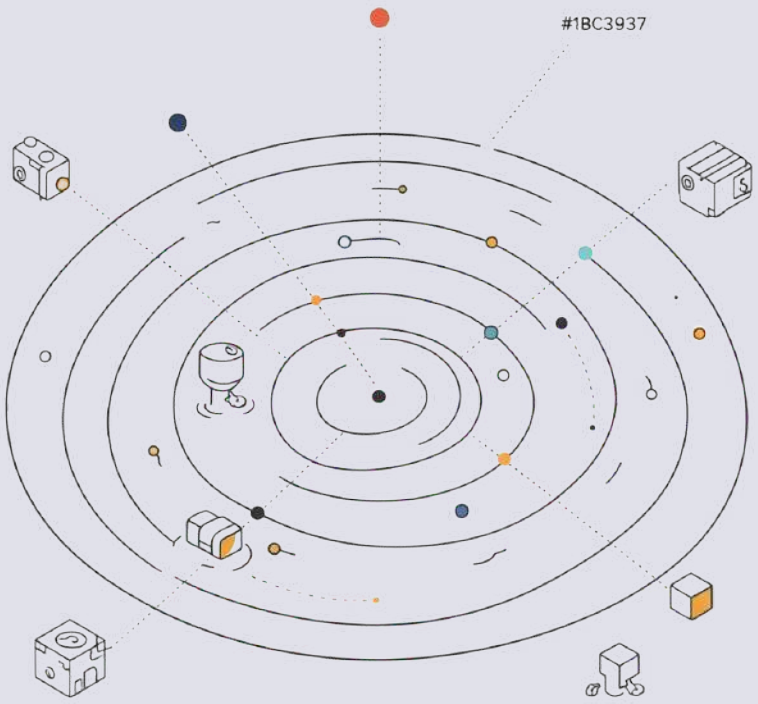
When $|r| < 1$

$$S = \frac{a}{1-r}$$

Diverges

When $|r| \geq 1$

Partial sums grow without bound.



Geometric Series: Examples

Example 1 — Converges

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$$

$a = 1,\ r = \frac{1}{2}$ – ratio inside $(-1, 1)$.

Example 2 — Converges

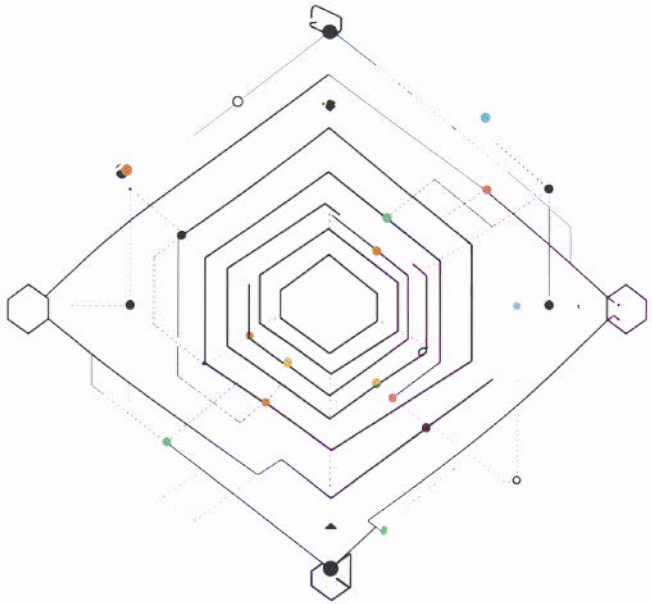
$$\sum_{n=1}^{\infty} 3 \cdot \left(\frac{2}{3}\right)^n = 3 \cdot \frac{\frac{2}{3}}{1-\frac{2}{3}} = 2$$

$a = 3,\ r = \frac{2}{3}.$

Example 3 — Diverges

$$\sum_{n=0}^{\infty} 2^n \quad \text{diverges}$$

Because $|r| = 2 > 1.$



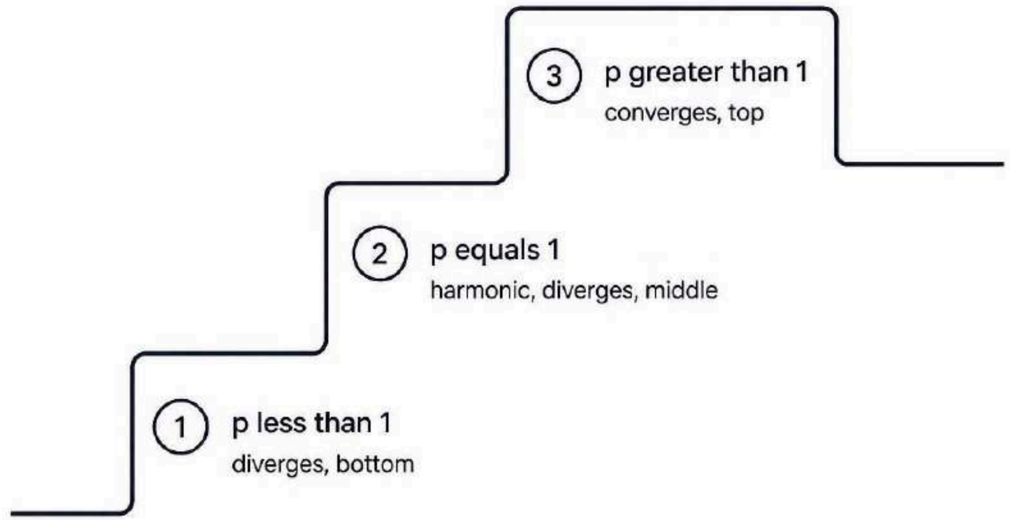
The p-Series

The **p-series** is defined as:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad p > 0$$

The single parameter p completely determines behavior:

Converges	Diverges
$p > 1$	$p \leq 1$



p-Series: Key Examples

Harmonic Series — Diverges

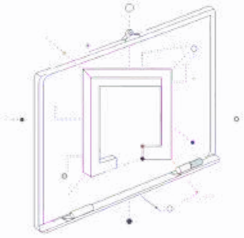
$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad p = 1$$

The classic example: terms go to 0, yet the series diverges.

Basel Problem — Converges

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad p = 2$$

A famous result: the sum converges to an exact value involving π.

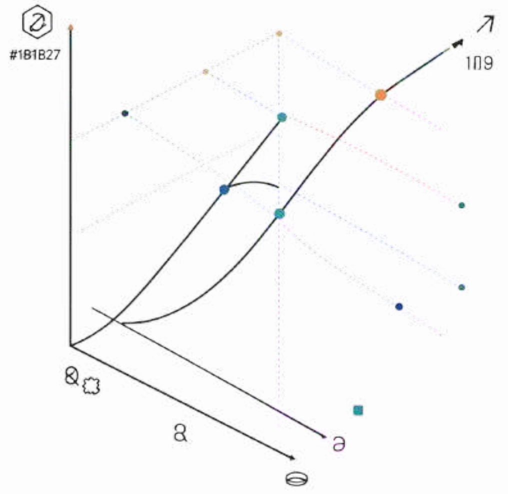


Square Root Series — Diverges

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}, \quad p = \frac{1}{2}$$

Since $p = \frac{1}{2} \leq 1$, this series diverges.

Integral Test



When a series resembles a function you can integrate, the **Integral Test** connects series and integrals:

> If $f(n) = a_n$ is **positive**, **continuous**, and **decreasing**, then $\sum a_n$ and $\int_1^\infty f(x)\,dx$ either *both converge* or *both diverge*.

Why it proves the p-series rule:

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ converges} \iff p > 1$$

This integral is exactly what underlies the p-series convergence criterion.

Comparison Tests

Direct Comparison

If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ **converges**.

If $a_n \geq b_n \geq 0$ and $\sum b_n$ diverges, then $\sum a_n$ **diverges**.

Limit Comparison

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $L > 0$ and finite, then $\sum a_n$ and $\sum b_n$ **behave the same** — both converge or both diverge.

Both tests work best when you can identify a **known benchmark series** (geometric or p-series) to compare against.

Absolute vs. Conditional Convergence

Absolutely Convergent

$$\sum |a_n| \text{ converges}$$

The series converges even when all signs are made positive. This is the **stronger** condition.

Conditionally Convergent

$$\sum a_n \text{ converges, but } \sum |a_n| \text{ diverges}$$

The alternating signs are *essential* to the convergence.

Key rule: Absolute convergence always implies convergence — but convergence does *not* imply absolute convergence.