

Chapter - 1

Vector Spaces

Vector Space

Let $(F, +, \cdot)$ be a field. Let V be a non empty set whose elements are vectors. Then V is a vector space over the field F , if the following conditions are satisfied:

1. $(V, +)$ is an abelian group

(i) **Closure property:** V is closed with respect to addition i.e.,

$$\alpha \in V, \beta \in V \Rightarrow \alpha + \beta \in V$$

(ii) **Associative:** $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma, \forall \alpha, \beta, \gamma \in V$

(iii) **Existence of identity:** $\exists$ an elements $0 \in V$ (zero vector) such that

$$\alpha + 0 = \alpha, \forall \alpha \in V$$

(iv) **Existence of inverse:** To every vector α in V can be associated with a unique vector $-\alpha$ in V called the additive inverse i.e.,

$$\alpha + (-\alpha) = 0$$

(v) **Commutative:** $\alpha + \beta = \beta + \alpha, \forall \alpha, \beta \in V$

2. V is closed under scalar multiplication i.e.,

$$a \in F, \alpha \in V \Rightarrow a\alpha \in V$$

3. Multiplication and addition of vector is a distributive property i.e.,

(i) $a(\alpha + \beta) = a\alpha + a\beta, \forall a \in F, \alpha, \beta \in V$

(ii) $(a + b)\alpha = a\alpha + b\alpha, \forall a, b \in F, \alpha \in V$

(iii) $(ab)\alpha = a(b\alpha), \forall a, b \in F, \alpha \in V$

(iv) $1 \cdot \alpha = \alpha, \forall \alpha \in V$ and 1 is the unity element in F .

Example 1. The vector space of all ordered n -tuples over a field F .

Proof. Let F be a field. An ordered set $\alpha = (a_1, a_2, \dots, a_n)$ of n -elements in F is called an n -tuples over F . Let V be the all ordered n -tuple over F . Let $V = \{(a_1, a_2, \dots, a_n) : a_1, a_2, \dots, a_n \in F\}$. Now, we will prove that V is a vector space over the field F . For this we define two n -tuples, addition and multiplication of two n -tuples by a scalar as follows.

Equality of two n -tuples : Let $\alpha = (a_1, a_2, \dots, a_n)$ and $\beta = (b_1, b_2, \dots, b_n)$ of V . Then $(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n) \Rightarrow a_i = b_i, \forall i = 1, 2, \dots, n$.

Addition of n -tuples : we take

$$\alpha + \beta = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n), \forall \alpha = (a_1, a_2, \dots, a_n) \in V,$$

$$\beta = (b_1, b_2, \dots, b_n) \in V$$

Since $a_1 + b_1, a_2 + b_2, \dots, a_n + b_n$ are all elements of F , therefore, $\alpha + \beta \in V$ and thus V is closed with respect to addition of n -tuples. Scalar multiplication of n -tuples : we define.

$$a\alpha = (aa_1, aa_2, \dots, aa_n), \forall a \in F, \alpha = (a_1, a_2, \dots, a_n) \in V,$$

Since $aa_1, aa_2, \dots, aa_n$ are all elements of F , therefore $a\alpha \in V$ and thus V is closed w.r.t. multiplication of n -tuples.

Now, we shall show that V is a vector space for the above two compositions.

1. (i) Associative : Let $(c_1, c_2, \dots, c_n) = \gamma \in V$

$$\begin{aligned} \alpha + (\beta + \gamma) &= (a_1, a_2, \dots, a_n) + [(b_1, b_2, \dots, b_n) + (c_1, c_2, \dots, c_n)] \\ &= (a_1, a_2, \dots, a_n) + [b_1 + c_1, b_2 + c_2, \dots, b_n + c_n] \\ &= a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \dots, a_n + (b_n + c_n) \\ &= (a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \dots, (a_n + b_n) + c_n \\ &= [(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n)] + (c_1, c_2, \dots, c_n) \\ &= (\alpha + \beta) + \gamma \end{aligned}$$

(ii) Commutative: We have

$$\begin{aligned} \alpha + \beta &= (a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) \\ &= (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \\ &= (b_1 + a_1, b_2 + a_2, \dots, b_n + a_n) \\ &= (b_1, b_2, \dots, b_n) + (a_1, a_2, \dots, a_n) \\ &= \beta + \alpha \end{aligned}$$

(iii) Existence of Identity : Let $(0, 0, \dots, 0) \in V$ then, we have

$$\begin{aligned} \alpha + 0 &= (a_1, a_2, \dots, a_n) + (0, 0, \dots, 0) \\ &= (a_1 + 0, a_2 + 0, \dots, a_n + 0) \\ &= (a_1, a_2, \dots, a_n) = \alpha \end{aligned}$$

(iv) **Existence of Inverse :** If $\alpha = (a_1, a_2, \dots, a_n)$ then

$$-\alpha = (-a_1, -a_2, \dots, -a_n) \in V$$

Then we have

$$\begin{aligned}\alpha + (-\alpha) &= (a_1, a_2, \dots, a_n) + (-a_1, -a_2, \dots, -a_n) \\ &= (a_1 - a_1, a_2 - a_2, \dots, a_n - a_n) \\ &= (0, 0, \dots, 0)\end{aligned}$$

Hence V is an abelian group under addition.

2. (i) If $a \in F$ and $(a_1, a_2, \dots, a_n) = \alpha \in V$, $(b_1, b_2, \dots, b_n) = \beta \in V$ then,

$$\begin{aligned}a(\alpha + \beta) &= a[(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n)] \\ &= a[a_1 + b_1, a_2 + b_2, \dots, a_n + b_n] \\ &= a(a_1 + b_1), a(a_2 + b_2), \dots, a(a_n + b_n) \\ &= (aa_1 + ab_1, aa_2 + ab_2, \dots, aa_n + ab_n) \\ &= (aa_1, aa_2, \dots, aa_n) + (ab_1, ab_2, \dots, ab_n) \\ &= a(a_1, a_2, \dots, a_n) + a(b_1, b_2, \dots, b_n) = a\alpha + a\beta\end{aligned}$$

(ii) If $a, b \in F$ and $\alpha = (a_1, a_2, \dots, a_n) \in V$ then

$$\begin{aligned}(a + b)\alpha &= (a + b)(a_1, a_2, \dots, a_n) \\ &= [(a + b)a_1, (a + b)a_2, \dots, (a + b)a_n] \\ &= (aa_1 + ba_1, aa_2 + ba_2, \dots, aa_n + ba_n) \\ &= (aa_1, aa_2, \dots, aa_n) + (ba_1, ba_2, \dots, ba_n) \\ &= a(a_1, a_2, \dots, a_n) + b(a_1, a_2, \dots, a_n) \\ &= a\alpha + b\alpha\end{aligned}$$

(iii) If $a, b \in F$ and $\alpha = (a_1, a_2, \dots, a_n) \in V$ then

$$\begin{aligned}(ab)\alpha &= (ab)(a_1, a_2, \dots, a_n) \\ &= [(ab)a_1, (ab)a_2, \dots, (ab)a_n] \\ &= [a(ba_1), a(ba_2), \dots, a(ba_n)] \\ &= a(ba_1, ba_2, \dots, ba_n) \\ &= a[b(a_1, a_2, \dots, a_n)] \\ &= a(b\alpha)\end{aligned}$$

(iv) If 1 is the unity element of F and $\alpha = (a_1, a_2, \dots, a_n) \in V$ then

$$1 \cdot \alpha = 1 \cdot (a_1, a_2, \dots, a_n) = (a_1, a_2, \dots, a_n) = \alpha$$

Hence V is a vector space over the field F . The vector space of all ordered n -tuples over F will be denoted by $V_n(F)$.

Example 2. Prove that the set of all vectors in a plane over the field of real member is a vector space.

Proof: Let V be the set of all vectors in a plane and R be the field of real numbers.

Then we observe that

1. $(V, +)$ is abelian group:

(i) **Closure property :** Let $\alpha, \beta \in V \Rightarrow \alpha + \beta \in V$

(ii) **Commutative property :** Let $\alpha, \beta \in V$ then

$$\alpha + \beta = \beta + \alpha, \forall \alpha, \beta \in V$$

(iii) **Associative property :** $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma), \forall \alpha, \beta, \gamma \in V$

(iv) **Existence of Identity :** Zero vector O in V such that

$$\alpha + 0 = \alpha, \forall \alpha \in V$$

(v) **Existence of inverse :** If $\alpha \in V$, then the vector $-\alpha \in V$ such that

$$\alpha + (-\alpha) = 0$$

2. If $\alpha \in V$ and $m \in R$ (m is any scalar). Then the scalar multiplication

$$m\alpha \in V$$

3. Scalar multiplication and addition of vectors satisfy the following properties:

(i) $m(\alpha + \beta) = m\alpha + m\beta, \forall m \in R, \forall \alpha, \beta \in V$

(ii) $(m+n)\alpha = m\alpha + n\alpha, \forall m, n \in R, \forall \alpha \in V$

(iii) $(mn)\alpha = m(n\alpha), \forall m, n \in R, \forall \alpha \in V$

(iv) $1 \cdot \alpha = \alpha, \forall \alpha \in V$ and 1 is the unit element of field R .

Hence V is a vector space over the field R .

Example 3. Let R be the field of real numbers and let R_n be the set of all polynomials over the field R . Prove that R_n is a vector space over the field R . Where R_n is of degree at most n .

Solution. Here R_n is the set of polynomials of degree at most n over the field R . The set R_n is also includes the zero polynomial.

$$\text{So, } R_n = \{f(x) : f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n\},$$

$$\text{Where } a_0, a_1, a_2, \dots, a_n \in R$$

$$\text{If } f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

$$g(x) = b_0 + b_1x + b_2x^2 + \dots + b_nx^n$$

$$r(x) = c_0 + c_1x + c_2x^2 + \dots + c_nx^n$$

$$\text{Then, } f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_n + b_n)x^n \in R_n$$

Because it is also a polynomial of degree at most n over the field R .

Thus R_n is closed for addition of polynomials.

∴ Addition of polynomials is commutative as well as associative. The zero polynomial 0 is a member of R_n and is identity for addition of polynomials.

Again if $f(x) = a_0 + a_1x + \dots + a_nx^n \in R_n$

then $-f(x) = -a_0 - a_1x - a_2x^2 - \dots - a_nx^n \in R_n$

because it is also a polynomial of degree at most n over the field R .

We have $-f(x) + f(x) = \text{zero polynomial}$.

The polynomial $-f(x)$ is the inverse of $f(x)$ for addition of polynomials.

Hence R_n is an addition group for addition of polynomials.

Now we define scalar multiplication $cf(x)$ by the relation.

$$cf(x) = ca_0 + (ca_1)x + (ca_2)x^2 + \dots + (ca_n)x^n$$

Clearly $cf(x) \in R_n$ because it is also a polynomial of degree at most n over the field R . Then R_n is closed for scalar multiplication.

Now if $k_1, k_2 \in R$ and $f(x), g(x) \in R_n$ we have

$$k_1[f(x) + g(x)] = k_1f(x) + k_2g(x)$$

$$(k_1 + k_2)f(x) = k_1f(x) + k_2f(x)$$

and $(k_1k_2)f(x) = k_1[k_2f(x)]$ can be proved easily.

Also $1 \cdot f(x) = f(x), f(x) \in R_n$

Hence R_n is a vector space over the field R .

General properties of vector spaces: Let V be a vector space over field F and $\bar{0}$ be the zero vector of V . then,

$$(i) \quad a \cdot \bar{0} = \bar{0}, \forall a \in F$$

$$(ii) \quad a\alpha = \bar{0}, \forall a \in F, \alpha \in V$$

$$(iii) \quad a(-\alpha) = -(a\alpha), \forall a \in F, \alpha \in V$$

$$(iv) \quad (-a)\alpha = -(a\alpha), \forall a \in F, \alpha \in V$$

$$(v) \quad a(\alpha - \beta) = a\alpha - a\beta \quad \forall a \in F, \alpha, \beta \in V$$

$$(vi) \quad a = \bar{0} \Rightarrow a = 0 \text{ or } \alpha = \bar{0}$$

Proof :

$$\begin{aligned}
 \text{(i) We have, } a\bar{0} &= a(\bar{0} + \bar{0}) \\
 &= a(\bar{0}) + a(\bar{0}) \\
 \therefore \bar{0} + a\bar{0} &= a\bar{0} + a\bar{0}
 \end{aligned}$$

$\because V$ is an abelian group with respect to addition therefore by right cancellation law in V , we get $\bar{0} = a \cdot \bar{0}$

$$\begin{aligned}
 \text{(ii) } 0\alpha &= (0+0)\alpha \quad [\because 0+0=0 \in F, \text{ by distributive law}] \\
 &= 0\alpha + 0\alpha \\
 \bar{0} + 0\alpha &= 0\alpha + 0\alpha
 \end{aligned}$$

By right cancellation law in V , we get

$$\begin{aligned}
 \bar{0} &= 0\alpha \\
 \text{(iii) } a[\alpha + (-\alpha)] &= a\alpha + a(-\alpha) \\
 a \cdot \bar{0} &= a\alpha + a(-\alpha) \\
 \Rightarrow \bar{0} &= a\alpha + a(-\alpha) \\
 \Rightarrow a(-\alpha) &= -(a\alpha)
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv) Now, } [a + (-a)]\alpha &= a\alpha + (-a)\alpha \\
 \Rightarrow 0\alpha &= a\alpha + (-a)\alpha \\
 \Rightarrow \bar{0} &= a\alpha + (-a)\alpha \\
 (-a)\alpha &\text{ is the additive inverse of } a\alpha \\
 \Rightarrow (-a)\alpha &= -(a\alpha)
 \end{aligned}$$

$$\begin{aligned}
 \text{(v) We have, } a(\alpha - \beta) &= a[\alpha + (-\beta)] \\
 &= a\alpha + a(-\beta) \\
 &= a\alpha + [-(a\beta)] \quad [\because a(-\beta) = -(a\beta)] \\
 &= a\alpha - a\beta
 \end{aligned}$$

(vi) Let $a\alpha = \bar{0}$ then we have to prove that either $a = 0$ or $\alpha = \bar{0}$.

Let $a\alpha = \bar{0}$ and $a \neq 0 \in F$, so $\hat{a}$ exists

$$\begin{aligned}
 \text{Then } \hat{a}(a\alpha) &= \hat{a} \cdot \bar{0} \\
 \Rightarrow (\hat{a}a)\alpha &= \bar{0} \\
 \Rightarrow 1 \cdot \alpha &= \bar{0}
 \end{aligned}$$

$$\Rightarrow \alpha = \bar{0}$$

So when $a \neq 0$ then $\alpha = \bar{0}$

again let $a\alpha = \bar{0}$ and we have to prove that $a = 0$. Suppose $a \neq 0$ then $\hat{a}$ exists.

$$\text{Now, } a\alpha = \bar{0}$$

$$\hat{a}(a\alpha) = \hat{a} \cdot \bar{0}$$

$$(\hat{a}a)\alpha = \bar{0}$$

$$\Rightarrow 1 \cdot \alpha = \bar{0}$$

$$\Rightarrow \alpha = \bar{0}$$

Which is a contradiction that α must be a zero vector. Therefore $a = 0$

Hence $a\alpha = \bar{0}$ then either $a = 0$ or $\alpha = \bar{0}$.

Vector Subspace: Let V be a vector space over the field F and W be a subset of V . then W is said to be vector subspace of V if W is also a vector space with scalar multiplication and vector addition over the field F as V .

Some basic theorems of vector subspaces

Theorem 1: The necessary and sufficient condition for a non empty subset W of a vector space V to be subspace of V is that W is closed under vector addition and scalar multiplication.

Proof: Condition is necessary. Let V be a vector space and W be subspace of V over the same field F . Since W is vector sub space of V , so it is also a vector space under vector addition and scalar multiplications so it is closed. Hence condition is necessary.

The condition is sufficient: Let V be a vector space over field F and W be a non empty subset of V , such that W is closed under vector addition and scalar multiplication then we have to prove that W is subspace of V . For this we will prove that it is a vector space over field itself.

Let $\alpha \in W$, if 1 is the unit element of F then $-1 \in F$. Now W is closed under scalar multiplication. Therefore,

$$(-1) \in F, \alpha \in W \Rightarrow (-1)\alpha \in W \Rightarrow -(1\alpha) \in W \Rightarrow -\alpha \in W$$

Thus, the additive inverse of each element of W is also in W . Now, W is closed under vector addition. Therefore

$$\alpha \in W \Rightarrow -\alpha \in W \Rightarrow \alpha + (-\alpha) \in W \Rightarrow \bar{0} \in W$$

Where $\bar{0}$ is the zero vector of V . Hence the zero vector of V is also the zero vector of W . Since $W \subseteq V$ therefore vector addition will be commutative as well as associative in W . Hence W is an abelian group with respect to vector addition. Also it is given that W is closed under scalar multiplication. The remaining properties of a vector space will hold in W , since they hold in V of which W is a subset. Hence W is itself a vector space with respect to vector addition and scalar multiplication as in V , so W is subspace of V . Hence condition is sufficient.

Theorem 2: The necessary and sufficient condition for a non empty subset W of a vector space V to be a subspace of V is

$$a, b \in F, \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$$

Proof: The condition is necessary: let V be a vector space over field F and W is subspace of V ; then by the definition of subspace, W is a vector space over field F itself as V .

So, $\forall a \in F, \alpha \in W \Rightarrow a\alpha \in W$

$$\forall b \in F, \beta \in W \Rightarrow b\beta \in W$$

$$\forall a\alpha \in W, b\beta \in W, \text{ by vector addition in } W, a\alpha + b\beta \in W$$

So condition is necessary.

The sufficient condition: Suppose V is a vector space over field F and W is nonempty subset of V such that $\forall a, b \in F, \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$ then we have to show that W is subspace of V , for this we will show that W is a vector space itself as V .

$$\therefore a\alpha + b\beta \in W$$

$$\text{Put } a = b = 1 \in F$$

$$\Rightarrow 1 \cdot \alpha + 1 \cdot \beta \in W$$

$$\Rightarrow \alpha + \beta \in W, \forall \alpha, \beta \in W$$

So W is closed under vector addition.

Now taking $a = 0, b = 0$, we see that if

$$\alpha \in W \text{ then}$$

$$0\alpha + 0\beta \in W$$

$$\bar{0} \in W$$

Thus the zero vector of V belongs to W . It will also be the zero vector of W .

Now again $1 \in F, -1 \in F$

$$\begin{aligned}
\text{Taking} \quad & a = -1, b = 0 \\
\text{We get} \quad & -1 \cdot \alpha + 0\bar{0} \in W \\
\Rightarrow \quad & -\alpha \in W
\end{aligned}$$

Thus the additive inverse of each elements of W is also in W .

Now taking $\beta = \bar{0}$, we see that if $a, b \in F$ and $\alpha \in W$, then $a\alpha + b\bar{0} \in W$ i.e., $a\alpha \in W$.

Thus W is closed under scalar multiplication. The remaining properties of a vector space will hold in W since they hold in V of which W is a subset.

Hence W is vector space itself. So by the definition W will be sub space of V .

Theorem 3. The necessary and sufficient conditions for a non-empty subset W of a vector space $V(F)$ to be a subspace of V are

- (i) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$
- (ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

Proof: As theorem 2.

Theorem 4. V be a vector space and W is non empty subset of V then W will be sub space of V if and only if $\forall \alpha, \beta \in W, a \in F \Rightarrow a\alpha + \beta \in W$.

Proof: As theorem 2.

Examples on Vector Sub Spaces

Example 1. The set W of ordered trails $(k_1, k_2, 0)$ where $k_1, k_2 \in F$ is a subspace of $V_3(F)$.

Solution. Let $\alpha = (k_1, k_2, 0)$, and $\beta = (l_1, l_2, 0)$ be any two element of W . Where $k_1, k_2, l_1, l_2 \in F$. If a, b be any two elements of F , we have

$$\begin{aligned}
a\alpha + b\beta &= a(k_1, k_2, 0) + b(l_1, l_2, 0) \\
&= (ak_1, ak_2, 0) + (bl_1, bl_2, 0) \\
&= (ak_1 + bl_1, ak_2 + bl_2, 0)
\end{aligned}$$

$$\because ak_1 + bl_1, ak_2 + bl_2 \in F \quad \text{so}$$

$$a\alpha + b\beta \in W$$

Hence W is a subspace of $V_3(F)$.

Example 2. Prove that the set of all solution (l, m, n) of the equation $l+m+2n=0$ is a subspace of the vector space $V_3(R)$

Solution. Let $W = \{(l, m, n) : l, m, n \in R \text{ and } l + m + 2n = 0\}$

To prove that W is a subspace of $V_3(R)$ or R^3 .

Let $\alpha = (l_1, m_1, n_1)$ and $\beta = (l_2, m_2, n_2)$ be any two elements of W . Then

$$l_1 + m_1 + 2n_1 = 0$$

$$l_2 + m_2 + 2n_2 = 0$$

If a, b be any two elements of R , we have

$$\begin{aligned} a\alpha + b\beta &= a(l_1, m_1, n_1) + b(l_2, m_2, n_2) \\ &= (al_1, am_1, an_1) + (bl_2, bm_2, bn_2) \\ &= (al_1 + bl_2, am_1 + bm_2, an_1 + bn_2) \end{aligned}$$

$$\begin{aligned} \text{Now } (al_1 + bl_2) + (am_1 + bm_2) + 2(an_1 + bn_2) \\ &= a(l_1 + m_1 + 2n_1) + b(l_2 + m_2 + 2n_2) \\ &= a \cdot 0 + b \cdot 0 = 0 \end{aligned}$$

$$\text{So } a\alpha + b\beta = (al_1 + bl_2, am_1 + bm_2, an_1 + bn_2) \in W$$

Thus, $\alpha, \beta \in W$ and $a, b \in R \Rightarrow a\alpha + b\beta \in W$.

Hence W is a subspace of $V_3(R)$.

Example 3. If V is a vector space of all real valued continuous functions over the field of real numbers R , then show that the set W of solutions of the differential equation.

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0 \text{ is a subspace of } V.$$

Solution. We have $W = \left\{ y : \frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0 \right\}$

It is clear that $y = 0$ satisfies the given differential equation and as such it belongs to W and thus $W \neq \Phi$.

Now let $y_1, y_2 \in W$ then

$$\frac{d^2 y_1}{dx^2} - 7 \frac{dy_1}{dx} + 12y_1 = 0 \quad \dots (i)$$

$$\frac{d^2 y_2}{dx^2} - 7 \frac{dy_2}{dx} + 12y_2 = 0 \quad \dots (ii)$$

Let $a, b \in R$. If W is to be subspace then we should show that $ay_1 + by_2$ also belongs to W i.e. It is a solution of the given differential equation. We have

$$\begin{aligned} & \frac{d^2}{dx^2} (ay_1 + by_2) - 7 \frac{d}{dx} (ay_1 + by_2) + 12(ay_1 + by_2) \\ &= a \cdot \left(\frac{d^2 y_1}{dx^2} - 7 \frac{dy_1}{dx} + 12y_1 \right) + b \left(\frac{d^2 y_2}{dx^2} - 7 \frac{dy_2}{dx} + 12y_2 \right) \\ &= a \cdot 0 + b \cdot 0 \end{aligned}$$

Thus $ay_1 + by_2$ is a solution of the given differential equation and so it belongs to W . Hence W is a subspace of V .

Algebra of subspaces

Theorem 1. The intersection of any two subspaces W_1 and W_2 of a vector space V (f) is also a subspace of V (f).

Proof : Let V be a vector space over field F and W_1, W_2 are two subspaces of V . It is clear that $\bar{0} \in W_1$ and $\bar{0} \in W_2$ so $W_1 \cap W_2 \neq \Phi$

Let, $\alpha, \beta \in W_1 \cap W_2$ and $a, b \in F$

$$\alpha \in W_1 \cap W_2 \Rightarrow \alpha \in W_1 \text{ and } \alpha \in W_2$$

$$\beta \in W_1 \cap W_2 \Rightarrow \beta \in W_1 \text{ and } \beta \in W_2$$

$\therefore W_1$ is subspace of V so

$$\forall a, b \in F, \alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$$

$$\forall a, b \in F, \alpha, \beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2$$

So, $a\alpha + b\beta \in W_1 \cap W_2$

So, $W_1 \cap W_2$ is subspace of V .

Theorem 2. The union of two subspaces is a subspace if and only if one is contained in the other .

Proof: Suppose W_1 and W_2 are two subspaces of V .

Condition is necessary: Let $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then we will prove that $W_1 \cup W_2$ will be subspace of V .

If $W_1 \subseteq W_2 \Rightarrow W_1 \cup W_2 = W_2$ and if $W_2 \subseteq W_1 \Rightarrow W_1 \cup W_2 = W_1$.

But W_1 and W_2 both are subspace of V , so $W_1 \cup W_2$ will be subspace of V .

Condition is sufficient: Let W_1 and W_2 be two subspaces of V such that $W_1 \cup W_2$ be also subspace of V , we have to show that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. Let us assume that W_1 is not a sub set of W_2 and W_2 is also not a subset of W_1 .

$\therefore W_1 \subseteq W_2 \Rightarrow \exists \alpha \in W_1$ such that $\alpha \notin W_2$

and $W_2 \subseteq W_1 \Rightarrow \exists \beta \in W_2$ such that $\beta \notin W_1$

But, $\alpha \in W_1 \cup W_2$

and $\beta \in W_1 \cup W_2$

But $W_1 \cup W_2$ is subspace of V so

$$\alpha + \beta \in W_1 \cup W_2$$

$$\Rightarrow \alpha + \beta \in W_1 \text{ or } \alpha + \beta \in W_2$$

If $\alpha + \beta \in W_1$ and $\alpha \in W_1$, $-\alpha \in W_1$

$$\Rightarrow (\alpha + \beta) - \alpha \in W_1 \Rightarrow \beta \in W_1$$

Which is a contradiction, again if

$$\Rightarrow \alpha + \beta \in W_2 \text{ and } -\beta \in W_2$$

$$\Rightarrow (\alpha + \beta) + (-\beta) \in W_2$$

$$\Rightarrow \alpha \in W_2$$

Again we get a contradiction. Hence either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Theorem 3. Intersection of any family of subspaces of a vector space is a subspace.

Proof: As above

Linear combination : Let V be a vector space if $\alpha_1, \alpha_2, \dots, \alpha_n \in V$ then any vector $\alpha \in V$

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$ where $a_i \in F$ is called a linear combination of the vectors $\alpha_1, \alpha_2, \dots, \alpha_n$.

Linear span : Let V be a vector space and S be any non empty subset of V . Then the linear span of S is the set of all linear combinations of finite sets of elements of S and is denoted by $L(S)$. Thus we have

$$L(S) = \{a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n, \alpha_i \in V, a_i \in F\}$$

Linear dependence and linear independence. Let V be a vector space over field F . A finite set $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is said to be linearly dependent if

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = \bar{0}$$

Where $\alpha_i \in V, a_i \in F$, and all a_i s may not zero. There will be minimum one $a_i \neq 0$.

A finite set $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is said to be linearly independent if $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = \bar{0}$

Where α_i 's $\in V$ and a_i 's $\in F$ and all a_i 's = 0

Any infinite set of vectors of V is said to be linearly independent if its every finite subset is linearly independent, otherwise it is linearly dependent.

Example 4. Show that the vector $(1, 2, 0), (0, 3, 1), (-1, 0, 1)$ forms linearly independent set over field R .

Solutions. Let, $S = \{\alpha_1, \alpha_2, \alpha_3\}$

Where, $\alpha_1 = (1, 2, 0), \alpha_2 = (0, 3, 1), \alpha_3 = (-1, 0, 1)$

Let $a_1, a_2, a_3 \in F$ such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$$

Then S will be linearly independent if all $a_1 = a_2 = a_3 = 0$

Now $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$

$$a_1(1, 2, 0) + a_2(0, 3, 1) + a_3(-1, 0, 1) = (0, 0, 0)$$

$$(a_1, 2a_1, 0) + (0, 3a_2, a_2) + (-a_3, 0, a_3) = (0, 0, 0)$$

$$(a_1 - a_3, 2a_1 + 3a_2, a_2 + a_3) = (0, 0, 0)$$

$$a_1 + 0a_2 - a_3 = 0$$

$$2a_1 + 3a_2 + 0a_3 = 0$$

$$0a_1 + a_2 + a_3 = 0$$

Coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}, |A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1[3] - 0 - 1[2]$$

$$|A| = 1 \neq 0$$

$\therefore$ Rank $A = 3$, hence there will be only one solution $a_1 = a_2 = a_3 = 0$

Hence $S = \{\alpha_1, \alpha_2, \alpha_3\}$ is linearly independent.

Example 5. Show that $S = \{\alpha_1, \alpha_2, \alpha_3\}$ is linearly dependent over field $\mathbf{R}$. Where $\alpha_1 = (1, 3, 2), \alpha_2 = (1, -7, -8), \alpha_3 = (2, 1, -1)$.

Proof: Let $a_1, a_2, a_3 \in \mathbf{R}$

$$\text{Now, } a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$$

$$\Rightarrow a_1(1, 3, 2) + a_2(1, -7, -8) + a_3(2, 1, -1) = (0, 0, 0)$$

$$\Rightarrow (a_1 + a_2 + 2a_3, 3a_1 - 7a_2 + a_3, 2a_1 - 8a_2 - a_3) = (0, 0, 0)$$

$$a_1 + a_2 + 2a_3 = 0$$

$$3a_1 - 7a_2 + a_3 = 0$$

$$2a_1 - 8a_2 - a_3 = 0$$

Coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -7 & 1 \\ 2 & -8 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 3 & -7 & 1 \\ 2 & -8 & -1 \end{vmatrix}$$

$$= 1[7+8] - 1[-3-2] + 2[-24+14]$$

$$= 15 + 5 - 20 = 0$$

$$|A| = 0$$

So rank of $A < 3$

Rank of $A < \text{number of unknowns}$

So there is minimum one $a_i \neq 0$

So $S = \{\alpha_1, \alpha_2, \alpha_3\}$ is linearly dependent.

Example 6. If α, β, γ are linearly independent vectors of $V(f)$ where F is any sub field of complex numbers then prove that $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ are also linearly independent.

Solution. Let a_1, a_2, a_3 be scalar then

$$a_1(\alpha + \beta) + a_2(\beta + \gamma) + a_3(\gamma + \alpha) = \bar{0}$$

$$(a_1 + a_3)\alpha + (a_1 + a_2)\beta + (a_2 + a_3)\gamma = \bar{0}$$

...(i)

But α, β, γ are linearly independent. Therefore (i) implies

$$a_1 + 0a_2 + a_3 = 0$$

$$a_1 + a_2 + 0a_3 = 0$$

$$0a_1 + a_2 + a_3 = 0$$

The coefficient matrix A of these equations is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$|A| = 1[0] + 0 + 1[1 - 0] = 1 \neq 0$$

Rank $A = 3 =$ number of unknowns

There is only one solution

$$a_1 = a_2 = a_3 = 0$$

So $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ are also linearly independent.

Basic of a vector space : A subset $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of a vector space $V(f)$ is said to be a basis of $V(f)$ if

(i) S consists of linearly independent vectors *i.e.*,

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$$

all a_i 's are zero, $\forall a_i \in F, \alpha_i \in V$

(ii) $L(S) = V(f)$ *i.e.*, every element of V can be written as linear combination of element of S .

Example 7. Show $S = \{(1, 2, 1), (2, 1, 0), (1, -1, 2)\}$ forms a basis of R^3 .

Proof : Since $\dim R^3 = 3$

So $L(S) = V(R^3)$

Now we only to prove that S is linearly independent

Let $a_1, a_2, a_3 \in F$ such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$$

We will prove that $a_1 = a_2 = a_3 = 0$

Now $a_1(1, 2, 1) + a_2(2, 1, 0) + a_3(1, -1, 2) = (0, 0, 0)$

$$(a_1 + 2a_2 + a_3, 2a_1 + a_2 - a_3, a_1 + 0a_2 + 2a_3) = (0, 0, 0)$$

$$a_1 + 2a_2 + a_3 = 0$$

$$2a_1 + a_2 - a_3 = 0$$

$$a_1 + 0a_2 + 2a_3 = 0$$

Coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= 1[2 - 0] - 2[4 + 1] + 1[0 - 1]$$

$$= 2 - 10 - 1$$

$$\neq 0$$

So rank of $A = 3 =$ no of unknown

So there is only one solution $a_1 = a_2 = a_3 = 0$

So $S = \{\alpha_1, \alpha_2, \alpha_3\}$ is linearly independent

So S forms the basis of R^3 .

Example 8. Select a basis of $R^3(R)$ from the set $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ where

$$\alpha_1 = (1, -3, 2), \alpha_2 = (2, 4, 1), \alpha_3 = (3, 1, 3), \alpha_4 = (1, 1, 1)$$

Solution. If any three vectors in S are linearly independent, then they will form a basis of the vector space $R^3(R)$.

First we take $S_1 = \{\alpha_1, \alpha_2, \alpha_3\}$

For this we take $a_1, a_2, a_3 \in R$ such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$

$$\text{i.e., } a_1(1, -3, 2) + a_2(2, 4, 1) + a_3(3, 1, 3) = (0, 0, 0)$$

$$(a_1 + 2a_2 + 3a_3, -3a_1 + 4a_2 + a_3, 2a_1 + a_2 + 3a_3) = (0, 0, 0)$$

$$a_1 + 2a_2 + 3a_3 = 0$$

$$-3a_1 + 4a_2 + a_3 = 0$$

$$2a_1 + a_2 + 3a_3 = 0$$

Coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$|A| = 0$$

So rank of $A < 3$

i.e., rank of $A < \text{no. of unknowns}$

So $S_1 = \{\alpha_1, \alpha_2, \alpha_3\}$ are linearly dependent

Now we take $S_2 = \{\alpha_1, \alpha_2, \alpha_4\}$

Then we get

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ -3 & 4 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

So rank of $A = \text{No. of unknown}$

So $[\alpha_1, \alpha_2, \alpha_4]$ is linearly independent

So $S_2 = \{\alpha_1, \alpha_2, \alpha_4\}$ forms the basis of $R^3 (R)$.

Linear Transformation: Let $U(f)$ and $V(f)$ be two vector space over the same field F . $T: U \rightarrow V$ is said to be linear transformation if

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \quad \dots(i)$$

$$\forall \alpha, \beta \text{ in } U \text{ and } a, b \text{ in } F$$

in another way the properly (i) can be defined in two ways

$$(i) \quad T(\alpha + \beta) = T(\alpha) + T(\beta) \text{ and}$$

$$(ii) \quad T(a\alpha) = aT(\alpha), \forall a \in F, \alpha, \beta \in U$$

Example 9. The function $T: V_3(R) \rightarrow V_2(R)$ defined by $T(a, b, c) = (a, b), \forall a, b, c \in R$, is a linear transformation from $V_3(R)$ into $V_2(R)$.

Proof: If $T: V_3(R) \rightarrow V_2(R)$ will be linear transformation then

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$$

Let $\alpha = (a_1, b_1, c_1) \in V_3(R)$

$$\beta = (a_2, b_2, c_2) \in V_3(R) \text{ and } a, b \in R$$

$$\begin{aligned} \text{Now } a\alpha + b\beta &= a(a_1, b_1, c_1) + b(a_2, b_2, c_2) \\ &= (aa_1, ab_1, ac_1) + (ba_2, bb_2, bc_2) \\ &= (aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2) \end{aligned}$$

$$\begin{aligned} \text{L.H.S. } &= T(a\alpha + b\beta) \\ &= T(aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2) \\ &= (aa_1 + ba_2, ab_1 + bb_2), \text{ by def. of } T \\ &= (aa_1, ab_1) + (ba_2, bb_2) \\ &= a(a_1, b_1) + b(a_2, b_2) \\ &= aT(a_1, b_1, c_1) + bT(a_2, b_2, c_2) \\ &= aT(\alpha) + bT(\beta) = R.H.S. \end{aligned}$$

So T is linear transformation from $V_3(R) \rightarrow V_2(R)$

Range and Null space of a linear transformation

Let U and V be two vector spaces over the same field F and let T be a linear transformation from U into V .

Then the range of T is written as $R(T)$ and it is the set of all vectors β in V such that $\beta = T(\alpha)$ for some $\alpha \in U$.

$$\text{Range } T = \{T(\alpha) = \beta : \alpha \in U, \beta \in V\}$$

Null space of a linear transformation

Let U and V be two vector spaces over same field F and let T be a linear transformation from U into V . Then the null space of T is written as $N(T)$ and it is the set of all vectors α in U such that $T(\alpha) = \bar{0}$ for some $\alpha \in U$.

$$\text{That is } N(T) = \{\alpha \in U : T(\alpha) = \bar{0} \in V\}$$

It is to be noted that if we take T as vector space homomorphism of U into V , then the null space of T is also called the kernel of T .

Rank and nullity of a linear transformation

Let U and V be two vector spaces over the same field F and T be a linear transformation from U to V , with U as finite dimensional.

The rank of T is denoted by $\rho(T)$ and it is the dimension of the range of T i.e.,

$$\rho(T) = \dim R(T)$$

The nullity of T is denoted by $\nu(T)$ and it is the dimension of null space of T i.e., $\nu(T) = \dim N(T)$

Note: Let $T : U(F) \rightarrow V(F)$ be a linear transformation from U into V . Suppose that U is finite dimensional. Then $\text{rank } T + \text{nullity } T = \dim U$.

Example 10. Show that the mapping $T : V_2(R) \rightarrow V_3(R)$ defined as $T(a, b) = (a + b, a - b, b)$ is a linear transformation from $V_2(R) \rightarrow V_3(R)$. Find the range, rank, null space and nullity of T .

Solution: Let $\alpha = (a_1, b_1)$, $\beta = (a_2, b_2)$ be arbitrary elements of $V_2(R)$. Then $T : V_2(R) \rightarrow V_3(R)$ will be a linear transformation if

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta), \forall a, b \in R$$

$$\alpha, \beta \in V_2(R), a, b \in R \quad \text{then} \quad a\alpha + b\beta \in V_2(R)$$

$$\text{Now, } T(a\alpha + b\beta) = T[a(a_1, b_1) + b(a_2, b_2)]$$

$$= T[(aa_1, ab_1) + (ba_2, bb_2)]$$

$$= T[(aa_1 + ba_2, ab_1 + bb_2)]$$

$$= [(aa_1 + ba_2) + (ab_1 + bb_2), (aa_1 + ba_2) - (ab_1 + bb_2), (ab_1 + bb_2)]$$

$$= [a(a_1 + b_1) + b(a_2 + b_2), a(a_1 - b_1) + b(a_2 - b_2), (ab_1 + bb_2)]$$

$$= a(a_1 + b_1, a_1 - b_1, b_1) + b(a_2 + b_2, a_2 - b_2, b_2)$$

$$= aT(a_1, b_1) + bT(a_2, b_2)$$

$$= aT(\alpha) + bT(\beta)$$

So $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$

So T is a linear transformation from $V_2(R)$ into $V_3(R)$.

Now $\{(1,0),(0,1)\}$ is a basis for $V_2(R)$

We have, $T(1, 0) = (1 + 0, 1 - 0, 0) = (1, 1, 0)$

$$T(0, 1) = (0 + 1, 0 - 1, 0) = (1, -1, 1)$$

The vectors $T(1, 0)$, $T(0, 1)$ span the range of T .

Thus the range of T is sub space of $V_3(R)$ spanned by the vectors $(1, 1, 0)$ and $(1, -1, 1)$.

Now the vectors $(1,1,0), (1,-1,1) \in V_3(R)$ are linearly independent if $x, y \in R$,

Then

$$x(1,1,0) + y(1,-1,1) = (0,0,0)$$

$$\Rightarrow (x+y, x-y, y) = (0,0,0)$$

$$\Rightarrow x+y=0, x-y=0, y=0$$

$$\Rightarrow x=0, y=0$$

The vectors $(1, 1, 0)$, $(1, -1, 1)$ form a basis for range of T .

Hence $\text{rank } T = \dim \text{ of range of } T = 2$

$$\text{Nullity of } T = \dim \text{ of } V_2(R) - \text{rank of } T = 2 - 2 = 0$$

Null space of T must be the zero subspace of $V_2(R)$

Otherwise $(a, b) \in \text{null space of } T$

$$\Rightarrow T(a, b) = (0, 0, 0)$$

$$(a+b, a-b, b) = (0, 0, 0)$$

$$a+b=0$$

$$a-b=0$$

$$b=0$$

$$\Rightarrow a=0, b=0$$

$\therefore (0,0)$ is the only element of $V_2(R)$ which belong to null space of T .

$\therefore$ Null space of T is the zero subspace of $V_2(R)$.

Representation of transformation by matrices

Let U be an n -dimensional vector space over the field F and let V be an m -dimensional vector space over the field F .

We take two ordered basis

$$\beta = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \text{ and } \beta' = \{\beta_1, \beta_2, \dots, \beta_m\}$$

for U and V respectively

Let $T: U \rightarrow V$ be a linear operator: since T is completely determined by its action on the vectors α_j belonging to a basis for U . Each of the n vectors $T(\alpha_j)$ is uniquely expressible as a linear combination of $\beta_1, \beta_2, \dots, \beta_m$. For $j = 1, 2, \dots, n$.

$$\text{Then, } T(\alpha_j) = a_{1j}\beta_1 + a_{2j}\beta_2 + \dots + a_{mj}\beta_m = \sum_{i=1}^m a_{ij}\beta_i$$

The scalars $a_{1j}, a_{2j}, \dots, a_{mj}$ are the co-ordinates of $T(\alpha_j)$ in the ordered basis β' .

The $m \times n$ matrix whose j^{th} column ($j = 1, 2, \dots, n$) consists of these co-ordinates is called the matrix of the linear transformation T relative to the pair of ordered basis β and β' . It is denoted by the symbol $[T: \beta: \beta']$ or simply by $[T]$ if the basis is understood. Thus,

$$[T] = [T: \beta: \beta'] = \text{matrix of } T \text{ relative to ordered basis } \beta \text{ and } \beta' = [a_{ij}]_{m \times n}$$

$$\text{and } T(\alpha_j) = \sum_{i=1}^m a_{ij}\beta_i, \forall j = 1, 2, \dots, n$$

Example 11. Find the matrix of the linear transformation T on $V_3(R)$ defined as $T(x, y, z) = (2y + z, x - 4y, 3x)$ with respect to the ordered basis β and also with respect to the ordered basis β' where

- (i) $\beta = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
- (ii) $\beta' = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$

Solution.

- (i) We have

$$T(1, 0, 0) = (0, 1, 3) = 0(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1)$$

$$T(0, 1, 0) = (2 - 4, 0) = 2(1, 0, 0) - 4(0, 1, 0) + 0(0, 0, 1)$$

$$\text{and } T(0, 0, 1) = (1, 0, 0) = 1(1, 0, 0) + 0(0, 1, 0) + 0(0, 0, 1)$$

so by def of matrix of T , with respect to β , we have

$$[T]_{\beta} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

(ii) We have $T(1, 1, 1) = (3, -3, 3)$

We have to express $(3, -3, 3)$ as a linear combination of vectors in β' .

$$\text{Let } (a, b, c) = x_1(1, 1, 1) + y_1(1, 1, 0) + z_1(1, 0, 0)$$

$$= (x_1 + y_1 + z_1, x_1 + y_1, x_1)$$

$$x_1 + y_1 + z_1 = a, x_1 + y_1 = b, x_1 = c$$

$$\text{So } x_1 = c, y_1 = b - c, z_1 = a - b$$

For $(3, -3, 3)$, putting $a = 3, b = -3, c = 3$

$$x_1 = 3, y_1 = -6 \text{ and } z_1 = 6 \quad \dots (i)$$

$$\text{So, } T(1, 1, 1) = (3, -3, 3) = 3(1, 1, 1) - 6(1, 1, 0) + 6(1, 0, 0)$$

$$\text{Also, } T(1, 1, 0) = (2, -3, 3)$$

Putting $a = 2, b = -3$ and $c = 3$ in (i) we get

$$T(1, 1, 0) = (2, -3, 3) = 3(1, 1, 1) - 6(1, 1, 0) + 6(1, 0, 0)$$

$$\text{Similarly, } T(1, 0, 0) = (0, 1, 3)$$

$$\text{So, } a = 0, b = 1, c = 3$$

$$T(1, 0, 0) = (0, 1, 3) = 3(1, 1, 1) - 2(1, 1, 0) - 1(1, 0, 0)$$

$$\text{So, } [T]_{\beta'} = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

Practice Problems

1. Suppose R be the field of real numbers. Which of the following are subspace of $V_3(R)$:

(i) $\{(a, 2b, 3c) : a, b, c \in R\}$,

(ii) $\{(a, a, a) : a \in R\}$

(iii) $\{(a, b, c) : a, b, c \text{ are rational numbers}\}$

Ans. (i) and (ii)

2. In $V_3(R)$, where R is the field of real numbers, examines each of the following sets of vectors for linear independence/ dependence.

(i) $\{(2, 1, 2), (8, 4, 8)\}$

(ii) $\{(-1, 2, 1), (3, 0, -1), (-5, 4, 3)\}$

(iii) $\{(2, 3, 5), (4, 9, 25)\}$

(iv) $\{(1, 2, 1), (3, 1, 5), (-4, 7)\}$

Ans. (i) Dependent, (ii) Dependent (iii) Independent (iv) Dependent.

3. Show that the three vectors $(1, 1, -1)$, $(2, -3, 5)$ and $(-2, 1, 4)$ of R^3 are linearly independent.

4. Determine if the set $\{(2, -1, 0), (3, 5, 1), (1, 1, 2)\}$ is a basis of $V_3(R)$.

5. Show that the vectors $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 1)$, $\alpha_3 = (0, -3, 2)$ form a basis of $V_3(R)$. Express each of the standard basis vectors as a linear combination of $\alpha_1, \alpha_2, \alpha_3$.

6. Show that the set $\{(1, i, 0), (2i, 1, 1), (0, 1+i, 1-i)\}$ is a basis for $V_3(c)$.

7. Let $T: V_3(R) \rightarrow V_3(R)$ defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3).$$

What is the matrix of T in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 = (1, 1, 0)$, $\alpha_2 = (-1, 2, 1)$, $\alpha_3 = (2, 1, 1)$.

$$\mathbf{Ans.} \quad T = \frac{1}{4} \begin{bmatrix} 17 & 35 & 22 \\ -3 & 15 & -6 \\ 2 & -14 & 0 \end{bmatrix}$$