quantization rule in a refined way which will ... hypothesis. The standing wave patterns of the electron in an orbit are illustrated in Figure 2.2.

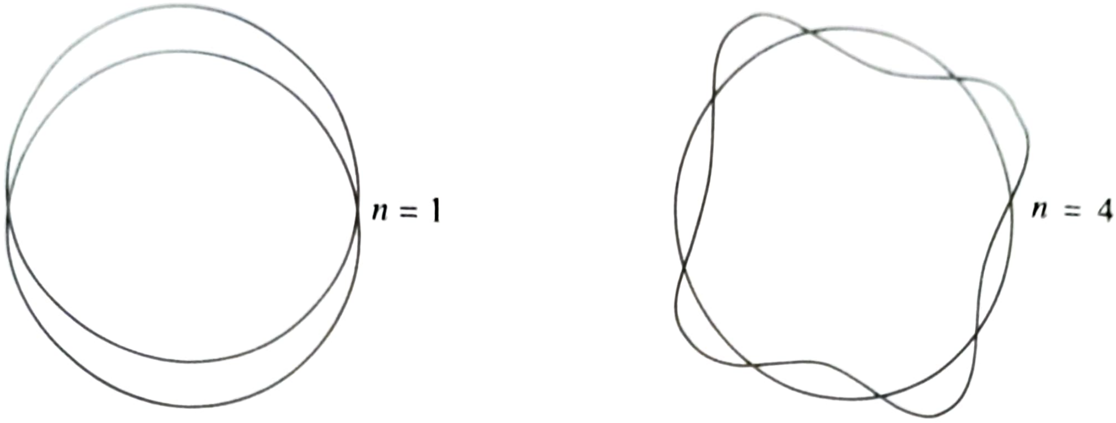

Figure 2.2 Standing wave patterns of an electron in an orbit for $n = 1$ and $n = 4$.

2.2 THE UNCERTAINTY PRINCIPLE

As per classical ideas, it is possible to determine all dynamical variables of a system to any desired degree of accuracy. This principle of determinism is the backbone of classical physics.

Position-momentum Uncertainty

The position of a plane wave is completely indeterminate as it is of infinite extent. Therefore, when waves are assigned to particles in motion an indeterminacy arises automatically in the formalism because an electron wave of definite frequency is not localized. Heisenberg analysed this indeterminacy and proposed that no two canonically conjugate quantities can be measured simultaneously. For the canonically conjugate variables x and p_x, mathematically, the principle is stated as

$$\Delta x \, \Delta p_x \cong h \tag{2.5}$$

The uncertainty relation can be illustrated by the *single-slit experiment* discussed below.

Single-slit experiment. Consider a beam of monoenergetic electrons of speed v_0 moving along the y-axis. Let us try to measure the position x of an electron and its velocity component v_x in the vertical direction (x-axis). To measure x, we insert a screen S_1 which has a slit of width Δx (Figure 2.3). If an electron passes through this slit, its vertical position is known to this accuracy. This can be improved by making the slit narrower.

As the electron has a wave nature, it will undergo diffraction at the slit giving the pattern as in Figure 2.3. Just at the time of reaching the slit, the velocity v_x of the electron is zero. The formation of the diffraction pattern shows that the electron has developed velocity component v_x after crossing the slit. For the first minimum, the theory of diffraction gives

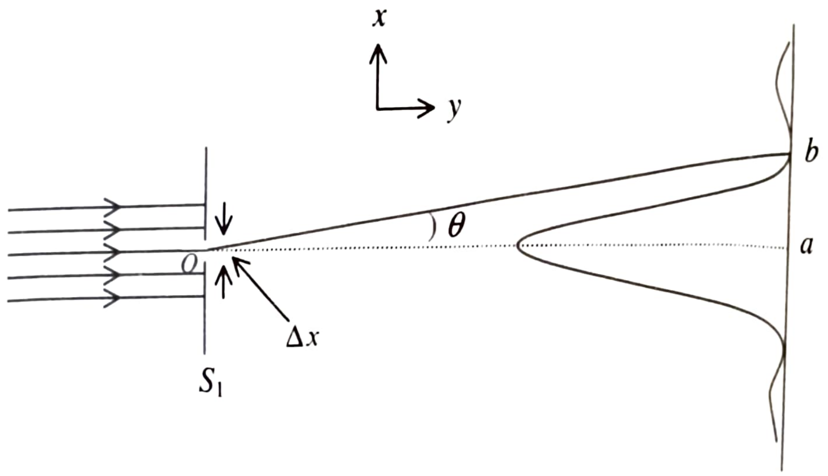

Figure 2.3 Illustration of Heisenberg uncertainty principle—single-slit experiment.

$$\sin\theta \cong \theta = \frac{\lambda}{\Delta x} \tag{2.6}$$

where λ is the wavelength of the electron beam. Let t be the time of transit from o to a and v_{xb} be the value of v_x at b. Then

$$v_0 t = oa \qquad \text{and} \qquad v_{xb} = \frac{ab}{t}$$

$$\tan\theta \cong \theta = \frac{ab}{oa} = \frac{t v_{xb}}{v_0 t} = \frac{v_{xb}}{v_0} \tag{2.7}$$

From Eqs. (2.6) and (2.7), we get

$$\frac{\lambda}{\Delta x} = \frac{v_{xb}}{v_0}$$

Replacing λ by h/mv_0 and taking v_{xb} as a rough measure of uncertainty Δv_x in v_x

$$\frac{h}{mv_0\Delta x} \cong \frac{\Delta v_x}{v_0} \quad \text{or} \quad \Delta x\Delta p_x \cong h \tag{2.8}$$

which is the desired relation. In the same way, we have

$$\Delta y\Delta p_y \cong h \quad \text{and} \quad \Delta z\Delta p_z \cong h \tag{2.9}$$

As the product of uncertainties is a universal constant, the more precisely we determine one variable, the less accurate is our determination of the other variable.

Before the introduction of the slit, the electrons travelling along the y-axis had the definite value of zero for p_x. By introducing the slit we measured the x-coordinate of the particles to an accuracy Δx, but this measurement introduced an uncertainty into the p_x values of the particles. Thus, the act of measurement introduced an uncontrollable disturbance in the system being measured which is a consequence of the wave particle duality.

Though the result is based on a particular experimental set-up, it is a very general one since it is independent of the particle mass and constants of the apparatus used. Heisenberg assumed it to be a fundamental law of nature. Therefore, the new mechanics we are looking for must abandon the deterministic model and allow only for probable values for dynamical variables.

The new mechanics was formulated independently by Werner Heisenberg (1925) and Erwin Schrödinger (1926). Heisenberg based his mechanics on matrix methods whereas Schrödinger used the idea of wave nature of electron. Introducing relativistic ideas, P.A.M. Dirac generalized quantum mechanics.

Uncertainty Relations for other Variables

Uncertainty relations can also be obtained for other pairs of canonically conjugate variables. For a free particle moving along x-axis, energy E is given by

$$E = \frac{p_x^2}{2m} \quad \text{or} \quad \Delta E = \frac{p_x}{m}\Delta p_x = v_x\,\Delta p_x$$

Therefore,

$$\Delta E = \frac{\Delta x}{\Delta t}\Delta p_x \quad \text{or} \quad \Delta E\,\Delta t = \Delta x\,\Delta p_x$$

Hence

$$\Delta E\,\Delta t \cong h \tag{2.10}$$

This equation indicates that if a system maintains a particular state for time Δt, its energy is uncertain at least by $\Delta E = h/\Delta t$.

The uncertainty for the pair of variables, component of angular momentum along the direction perpendicular to the plane of the orbit (L_z) of a particle and the angular position (ϕ) can be obtained as follows:

[illegible]

$$L_z^2 = 2IE$$

[illegible]

$$L_z \Delta L_z = I\Delta E$$

[illegible]

$$L_z = I\omega = I\frac{\Delta\phi}{\Delta t}$$

[illegible]

$$I\frac{\Delta\phi}{\Delta t}\Delta L_z = I\,\Delta E$$

[illegible] we have

$$\Delta\phi\,\Delta L_z = \Delta E\,\Delta t = h \tag{2.11}$$

These uncertainty relations are very useful in explaining number of observed phenomena which the classical physics failed. We shall now consider some of them.

Applications of Uncertainty Relations

Ground state energy of hydrogen atom. The classical expression for the total energy of the electron in the ground state is given by

$$E = \frac{p^2}{2m} - \frac{e^2}{4\pi\varepsilon_0 a} \tag{2.12}$$

where a is the radius of the first orbit. Let the uncertainty in the position of the electron Δx be of the order of a. More correctly, the product of uncertainties is given by $\hbar$. Therefore,

$$a\Delta p \cong \hbar \quad \text{or} \quad \Delta p \cong \frac{\hbar}{a} \tag{2.13}$$

Taking the momentum to be of the order of $\hbar/a$, we get

$$E = \frac{\hbar^2}{2ma^2} - \frac{e^2}{4\pi\varepsilon_0 a} \tag{2.14}$$

For the ground state, the energy E has to be minimum. For this, dE/da must be zero. Denoting the minimum radius by a_0, we have

$$\frac{dE}{da} = 0 = \frac{-\hbar^2}{ma_0^3} + \frac{e^2}{4\pi\varepsilon_0 a_0^2}$$

or

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2} \tag{2.15}$$

With this value of a, Eq. (2.14) becomes

$$E = -\frac{me^4}{(4\pi\varepsilon_0)^2 2\hbar^2} = -\frac{me^4}{8\varepsilon_0^2 h^2} \tag{2.16}$$

which is the ground state energy of the hydrogen atom.

Width of spectral lines. Spectral lines have finite width due to various factors. One such factor is the natural broadening which is a direct consequence of uncertainty principle. Atoms remain in the excited state for a finite time τ, called the *life time*; before making a transition. Hence there will be an uncertainty in time of the order of τ. Then

$$\tau\Delta E \cong \hbar \quad \text{or} \quad \Delta\nu \cong \frac{1}{2\pi\tau} \tag{2.17}$$

For most of the states, the life time $\tau = 10^{-8}$ s. Hence, $\Delta\nu \cong 10^8$ Hz. This spreading is experimentally observed when the pressure is very low.

Mass of meson. Yukawa proposed that nuclear forces are due to an exchange of mesons. Uncertainty principle may be used to derive a relation between mass of meson (m) and the range (r_0) of nuclear force. When one nucleon exerts force on the other, a meson is created. During transit, its position is uncertain by an amount r_0. Use of the uncertainty relation gives

$$r_0\Delta p \cong \hbar \quad \text{or} \quad \Delta p \cong \frac{\hbar}{r_0} \tag{2.18}$$

For a relativistic particle $p = mc$. Taking the uncertainty in momentum to be of this order, we obtain

$$mc \cong \frac{\hbar}{r_0} \quad \text{or} \quad m = \frac{\hbar}{r_0 c} \tag{2.19}$$

For $r_0 = 1.5 \times 10^{-13}$ cm, $m \cong 200\, m_e$, where m_e is the electron mass.

Nonexistence of electron in the nucleus. For an electron to exist inside a nucleus, the uncertainty in its position must be at least of the order of $2r_0$, r_0 being the radius of the nucleus. The uncertainty in the electron momentum is then

$$\Delta p \cong \frac{\hbar}{2r_0} \tag{2.20}$$

For a typical nucleus $r_0 \cong 10^{-14}$ m. Hence

$$\Delta p \cong \frac{1.055 \times 10^{-34}}{2 \times 10^{-14}} \cong 5.28 \times 10^{-21} \text{ kgms}^{-1}$$