Cauchy Criterion & Convergence Tests

Comprehensive analysis of infinite series, foundational convergence criteria, comparison techniques, and D'Alembert's ratio test with detailed worked examples.

Fundamentals of Infinite Series

What is an Infinite Series?

An infinite series is the sum of infinitely many terms expressed as:

$$\sum a_n = a_1 + a_2 + a_3 + ... + a_n + ...$$

The behavior of the series is determined by examining the limit of its sequence of partial sums $S_n = a_1 + a_2 + ... + a_n$ as n approaches infinity.

Core Convergence Definitions

Convergent Series: If $\lim_{n\to\infty} S_n$ exists and is a finite number S, the series converges to sum S.

Divergent Series: If $\lim_{n\to\infty} S_n$ does not exist or equals $\pm\infty$, the series diverges.

Necessary Condition: For $\sum a_n$ to converge, it is necessary that $\lim_{n\to\infty} a_n = 0$ (though not sufficient).

Cauchy's Criterion for Convergence

- **Statement:** An infinite series $\sum a_n$ converges if and only if for every $\varepsilon > 0$, there exists a positive integer N such that for all $n \geq N$ and any integer $p \geq 1$, the absolute difference of partial sums satisfies:

  > $$|a_{n+1} + a_{n+2} + \ldots + a_{n+p}| < \varepsilon$$

- **Significance:** This criterion is exceptionally powerful because it determines whether a series converges or diverges *without requiring the actual sum S* to be known beforehand.
- **Foundation:** It relies entirely on the completeness property of real numbers, establishing that a sequence of partial sums forms a Cauchy sequence in $\mathbb{R}$.

Direct Comparison Test

Theorem Statement

Let $\sum a_n$ and $\sum b_n$ be two series of positive terms such that $0 \leq a_n \leq b_n$ for all $n \geq k$ (where k is a fixed integer).

1. If the larger series **$\sum b_n$ converges**, then the smaller series **$\sum a_n$ also converges**.

2. If the smaller series **$\sum a_n$ diverges**, then the larger series **$\sum b_n$ also diverges**.

Standard Benchmark Series

To use comparison tests effectively, we compare against well-known standard series:

Geometric Series: $\sum r^n$ converges if $|r| < 1$ and diverges if $|r| \geq 1$.

p-Series: $\sum 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$ (e.g., Harmonic series diverges where $p = 1$).

Limit Comparison Test

- **Statement:** Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If the limit of their ratio exists as:

$$\lim_{n\to\infty} (a_n / b_n) = L$$

  where L is a finite number and $L > 0$, then **both series converge or both series diverge together**.

- **Special Cases:**
  - If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
  - If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

- **Utility:** Eliminates the need for strict inequalities ($a_n \le b_n$) required in direct comparison, making complex rational expressions much easier to evaluate.

Examples: Comparison Tests

Example 1: Direct Comparison

Test convergence of: $\sum 1 / (n^2 + 1)$

Solution: For all $n \geq 1$, we note that:

$$0 < 1 / (n^2 + 1) < 1 / n^2$$

Since the p-series $\sum 1/n^2$ converges ($p = 2 > 1$), by direct comparison test, the given series **converges**.

Example 2: Limit Comparison

Test convergence of: $\sum (2n + 1) / (3n^2 + n)$

Solution: Choose benchmark $b_n = n / n^2 = 1/n$. Compute limit:

$$\lim_{n\to\infty} [((2n+1)/(3n^2+n)) / (1/n)] = 2/3$$

Since $L = 2/3$ (finite and > 0) and $\sum 1/n$ diverges (harmonic series), the given series **diverges**.

D'Alembert's Ratio Test

- **Definition:** Developed by Jean le Rond d'Alembert, this test evaluates the limiting ratio of consecutive terms in a positive-term series $\sum a_n$.
- **Ratio Calculation:** Compute the limit:

$$L = \lim_{n \to \infty} (a_{n+1} / a_n)$$

- **Decision Criteria:**
  - If **$L < 1$**: The series converges absolutely.
  - If **$L > 1$** (or ∞): The series diverges.
  - If **$L = 1$**: The test is inconclusive (further tests needed).

Ratio Test: Factorial Example

Problem Statement

Determine the convergence of the infinite series:

$$\sum (3^n / n!)$$

Here, the general term is $a_n = 3^n / n!$ and the next term is $a_{n+1} = 3^{n+1} / (n + 1)!$.

Step-by-Step Solution

Set up the ratio and simplify:

$$a_{n+1} / a_n = [3^{n+1} / (n+1)!] \times [n! / 3^n]$$
$$= 3 / (n + 1)$$

Now take the limit as n approaches infinity:

$$L = \lim_{n\to\infty} [3 / (n + 1)] = 0$$

Since L = 0 < 1, the series **converges** by D'Alembert's ratio test.

Ratio Test: Polynomial / Exponential

Problem Statement

Test the convergence of the series:

$$\sum (n^2 / 2^n)$$

General term: $a_n = n^2 / 2^n$

Successive term: $a_{n+1} = (n + 1)^2 / 2^{n+1}$

Evaluation & Limit

Form the ratio:

$$a_{n+1} / a_n = [(n+1)^2 / 2^{n+1}] \times [2^n / n^2] = [(n+1)^2 / n^2] \times (1/2)$$

Computing the limit:

$$L = \lim_{n\to\infty} [(1 + 1/n)^2 \times (1/2)] = 1 \times (1/2) = 1/2$$

Since L = 1/2 < 1, the series **converges**.

Summary Matrix of Tests

Test Name	Applicable Series Type	Key Condition / Parameter	Conclusion when L / Test Fails
Cauchy Criterion	General Series	$\|S_{n+p} - S_n\| < \varepsilon$	Necessary and sufficient for convergence.
Comparison Test	Positive Term Series	$0 \leq a_n \leq b_n$	Inconclusive if bounds do not hold.
D'Alembert's Ratio	Positive Term Series	$L = \lim (a_{n+1} / a_n)$	Inconclusive if $L = 1$.

Key Takeaways & Applications

Analytical Rigor

Cauchy's criterion forms the backbone of mathematical analysis, guaranteeing convergence limits without summation bounds.

Comparative Power

Comparison tests provide quick convergence insight by mapping complex terms against standard geometric and p-series benchmarks.

Factorial & Power Growth

D'Alembert's ratio test excels when dealing with factorials, powers, and exponential terms where terms scale multiplicatively.

Questions & Discussion

Thank you for your attention! Open for any questions regarding convergence tests.