

Vector Analysis

1.1 Vector Algebra

1.1.1 Vector Operations

If you walk 4 miles due north and then 3 miles due east (Fig. 1.1), you will have gone a total of 7 miles, but you're *not* 7 miles from where you set out—you're only 5. We need an arithmetic to describe quantities like this, which evidently do not add in the ordinary way. The reason they don't, of course, is that **displacements** (straight line segments going from one point to another) have *direction* as well as *magnitude* (length), and it is essential to take both into account when you combine them. Such objects are called **vectors**: velocity, acceleration, force and momentum are other examples. By contrast, quantities that have magnitude but no direction are called **scalars**: examples include mass, charge, density, and temperature. I shall use **boldface** ($\mathbf{A}$, $\mathbf{B}$, and so on) for vectors and ordinary type for scalars. The magnitude of a vector $\mathbf{A}$ is written $|\mathbf{A}|$ or, more simply, A . In diagrams, vectors are denoted by arrows: the length of the arrow is proportional to the magnitude of the vector, and the arrowhead indicates its direction. *Minus* $\mathbf{A}$ ($-\mathbf{A}$) is a vector with the

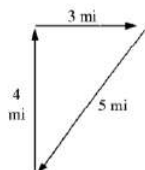


Figure 1.1

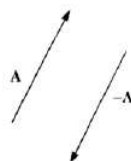


Figure 1.2

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CHAPTER 1. VECTOR ANALYSIS

same magnitude as $\mathbf{A}$ but of opposite direction (Fig. 1.2). Note that vectors have magnitude and direction but *not* location: a displacement of 4 miles due north from Washington is represented by the same vector as a displacement 4 miles north from Baltimore (neglecting, of course, the curvature of the earth). On a diagram, therefore, you can slide the arrow around at will, as long as you don't change its length or direction.

We define four vector operations: addition and three kinds of multiplication.

(i) **Addition of two vectors.** Place the tail of $\mathbf{B}$ at the head of $\mathbf{A}$; the sum, $\mathbf{A} + \mathbf{B}$, is the vector from the tail of $\mathbf{A}$ to the head of $\mathbf{B}$ (Fig. 1.3). (This rule generalizes the obvious procedure for combining two displacements.) Addition is *commutative*:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A};$$

3 miles east followed by 4 miles north gets you to the same place as 4 miles north followed by 3 miles east. Addition is also *associative*:

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

To subtract a vector (Fig. 1.4), add its opposite:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

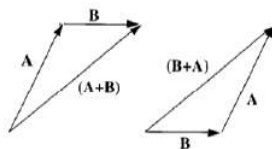


Figure 1.3

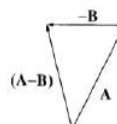


Figure 1.4

(ii) **Multiplication by a scalar.** Multiplication of a vector by a positive scalar a multiplies the *magnitude* but leaves the direction unchanged (Fig. 1.5). (If a is negative, the direction is reversed.) Scalar multiplication is *distributive*:

$$a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}.$$

(iii) **Dot product of two vectors.** The dot product of two vectors is defined by

$$\mathbf{A} \cdot \mathbf{B} \equiv AB \cos \theta, \quad (1.1)$$

where θ is the angle they form when placed tail-to-tail (Fig. 1.6). Note that $\mathbf{A} \cdot \mathbf{B}$ is itself a *scalar* (hence the alternative name **scalar product**). The dot product is *commutative*,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}.$$

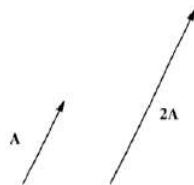


Figure 1.5

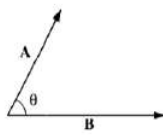


Figure 1.6

and *distributive*,

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}. \quad (1.2)$$

Geometrically, $\mathbf{A} \cdot \mathbf{B}$ is the product of A times the projection of $\mathbf{B}$ along $\mathbf{A}$ (or the product of B times the projection of $\mathbf{A}$ along $\mathbf{B}$). If the two vectors are parallel, then $\mathbf{A} \cdot \mathbf{B} = AB$. In particular, for any vector $\mathbf{A}$,

$$\mathbf{A} \cdot \mathbf{A} = A^2. \quad (1.3)$$

If $\mathbf{A}$ and $\mathbf{B}$ are perpendicular, then $\mathbf{A} \cdot \mathbf{B} = 0$.

Example 1.1

Let $\mathbf{C} = \mathbf{A} - \mathbf{B}$ (Fig. 1.7), and calculate the dot product of $\mathbf{C}$ with itself.

Solution:

$$\mathbf{C} \cdot \mathbf{C} = (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B},$$

or

$$C^2 = A^2 + B^2 - 2AB \cos \theta.$$

This is the **law of cosines**.

(iv) **Cross product of two vectors.** The cross product of two vectors is defined by

$$\mathbf{A} \times \mathbf{B} \equiv AB \sin \theta \hat{\mathbf{n}}, \quad (1.4)$$

where $\hat{\mathbf{n}}$ is a **unit vector** (vector of length 1) pointing perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$. (I shall use a hat (^) to designate unit vectors.) Of course, there are *two* directions perpendicular to any plane: "in" and "out." The ambiguity is resolved by the **right-hand rule**: let your fingers point in the direction of the first vector and curl around (via the smaller angle) toward the second; then your thumb indicates the direction of $\hat{\mathbf{n}}$. (In Fig. 1.8 $\mathbf{A} \times \mathbf{B}$ points *into* the page; $\mathbf{B} \times \mathbf{A}$ points *out* of the page.) Note that $\mathbf{A} \times \mathbf{B}$ is itself a *vector* (hence the alternative name **vector product**). The cross product is *distributive*,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C}), \quad (1.5)$$

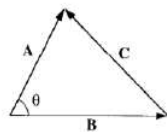


Figure 1.7

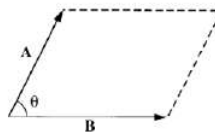


Figure 1.8

but *not commutative*. In fact,

$$(\mathbf{B} \times \mathbf{A}) = -(\mathbf{A} \times \mathbf{B}). \quad (1.6)$$

Geometrically, $|\mathbf{A} \times \mathbf{B}|$ is the area of the parallelogram generated by $\mathbf{A}$ and $\mathbf{B}$ (Fig. 1.8). If two vectors are parallel, their cross product is zero. In particular,

$$\mathbf{A} \times \mathbf{A} = 0$$

for any vector $\mathbf{A}$.

Problem 1.1 Using the definitions in Eqs. 1.1 and 1.4, and appropriate diagrams, show that the dot product and cross product are distributive,

a) when the three vectors are coplanar;

b) in the general case.

Problem 1.2 Is the cross product associative?

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \stackrel{?}{=} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}).$$

If so, *prove* it; if not, provide a counterexample.

1.1.2 Vector Algebra: Component Form

In the previous section I defined the four vector operations (addition, scalar multiplication, dot product, and cross product) in "abstract" form—that is, without reference to any particular coordinate system. In practice, it is often easier to set up Cartesian coordinates x, y, z and work with vector "components." Let $\hat{\mathbf{x}}, \hat{\mathbf{y}},$ and $\hat{\mathbf{z}}$ be unit vectors parallel to the $x, y,$ and z axes, respectively (Fig. 1.9(a)). The addition of two vectors is then done by adding their components along the axes.

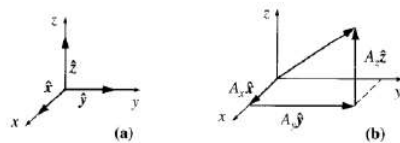


Figure 1.9

The numbers A_x , A_y , and A_z , are called **components** of $\mathbf{A}$; geometrically, they are the projections of $\mathbf{A}$ along the three coordinate axes. We can now reformulate each of the four vector operations as a rule for manipulating components:

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= (A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}) + (B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}) \\ &= (A_x + B_x) \hat{\mathbf{x}} + (A_y + B_y) \hat{\mathbf{y}} + (A_z + B_z) \hat{\mathbf{z}}.\end{aligned}\quad (1.7)$$

(i) **Rule:** To add vectors, add like components.

$$a\mathbf{A} = (aA_x) \hat{\mathbf{x}} + (aA_y) \hat{\mathbf{y}} + (aA_z) \hat{\mathbf{z}}.\quad (1.8)$$

(ii) **Rule:** To multiply by a scalar, multiply each component.

Because $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are mutually perpendicular unit vectors,

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1; \quad \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = 0.\quad (1.9)$$

Accordingly,

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= (A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}) \cdot (B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}) \\ &= A_x B_x + A_y B_y + A_z B_z.\end{aligned}\quad (1.10)$$

(iii) **Rule:** To calculate the dot product, multiply like components, and add.

In particular,

$$\mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2,$$

so

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}.\quad (1.11)$$

(This is, if you like, the three-dimensional generalization of the Pythagorean theorem.) Note that the dot product of $\mathbf{A}$ with any *unit* vector is the component of $\mathbf{A}$ along that direction (thus $\mathbf{A} \cdot \hat{\mathbf{x}} = A_x$, $\mathbf{A} \cdot \hat{\mathbf{y}} = A_y$, and $\mathbf{A} \cdot \hat{\mathbf{z}} = A_z$).

Similarly,¹

$$\begin{aligned}\hat{\mathbf{x}} \times \hat{\mathbf{x}} &= \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0, \\ \hat{\mathbf{x}} \times \hat{\mathbf{y}} &= -\hat{\mathbf{y}} \times \hat{\mathbf{x}} = \hat{\mathbf{z}}, \\ \hat{\mathbf{y}} \times \hat{\mathbf{z}} &= -\hat{\mathbf{z}} \times \hat{\mathbf{y}} = \hat{\mathbf{x}}, \\ \hat{\mathbf{z}} \times \hat{\mathbf{x}} &= -\hat{\mathbf{x}} \times \hat{\mathbf{z}} = \hat{\mathbf{y}}.\end{aligned}\quad (1.12)$$

Therefore,

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}) \times (B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}) \\ &= (A_y B_z - A_z B_y) \hat{\mathbf{x}} + (A_z B_x - A_x B_z) \hat{\mathbf{y}} + (A_x B_y - A_y B_x) \hat{\mathbf{z}}.\end{aligned}\quad (1.13)$$

This cumbersome expression can be written more neatly as a determinant:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.\quad (1.14)$$

(iv) **Rule:** To calculate the cross product, form the determinant whose first row is $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$, whose second row is $\mathbf{A}$ (in component form), and whose third row is $\mathbf{B}$.

Example 1.2

Find the angle between the face diagonals of a cube.

Solution: We might as well use a cube of side 1, and place it as shown in Fig. 1.10, with one corner at the origin. The face diagonals $\mathbf{A}$ and $\mathbf{B}$ are

$$\mathbf{A} = 1\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 1\hat{\mathbf{z}}; \quad \mathbf{B} = 0\hat{\mathbf{x}} + 1\hat{\mathbf{y}} + 1\hat{\mathbf{z}}.$$

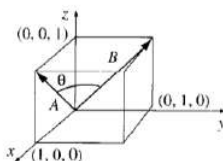


Figure 1.10

So, in component form,

$$\mathbf{A} \cdot \mathbf{B} = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1.$$

On the other hand, in "abstract" form,

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta = \sqrt{2}\sqrt{2} \cos \theta = 2 \cos \theta.$$

Therefore,

$$\cos \theta = 1/2, \quad \text{or} \quad \theta = 60^\circ.$$

Of course, you can get the answer more easily by drawing in a diagonal across the top of the cube, completing the equilateral triangle. But in cases where the geometry is not so simple, this device of comparing the abstract and component forms of the dot product can be a very efficient means of finding angles.

Problem 1.3 Find the angle between the body diagonals of a cube.

Problem 1.4 Use the cross product to find the components of the unit vector $\hat{\mathbf{n}}$ perpendicular to the plane shown in Fig. 1.11.

1.1.3 Triple Products

Since the cross product of two vectors is itself a vector, it can be dotted or crossed with a third vector to form a *triple product*.

(i) **Scalar triple product:** $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$. Geometrically, $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$ is the volume of the parallelepiped generated by $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, since $|\mathbf{B} \times \mathbf{C}|$ is the area of the base, and $|\mathbf{A} \cos \theta|$ is the altitude (Fig. 1.12). Evidently,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}), \quad (1.15)$$

for they all correspond to the same figure. Note that "alphabetical" order is preserved—in view of Eq. 1.6, the "nonalphabetical" triple products,

$$\mathbf{A} \cdot (\mathbf{C} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{B} \times \mathbf{A}),$$

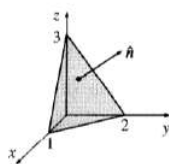


Figure 1.11

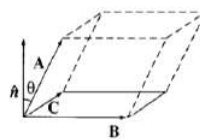


Figure 1.12

have the opposite sign. In component form,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \quad (1.16)$$

Note that the dot and cross can be interchanged:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$$

(this follows immediately from Eq. 1.15); however, the placement of the parentheses is critical: $(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$ is a meaningless expression—you can't make a cross product from a scalar and a vector.

(ii) **Vector triple product:** $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. The vector triple product can be simplified by the so-called **BAC-CAB** rule:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}). \quad (1.17)$$

Notice that

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = -\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = -\mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{B}(\mathbf{A} \cdot \mathbf{C})$$

is an entirely different vector. Incidentally, all *higher* vector products can be similarly reduced, often by repeated application of Eq. 1.17, so it is never necessary for an expression to contain more than one cross product in any term. For instance,

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}); \\ \mathbf{A} \times (\mathbf{B} \times (\mathbf{C} \times \mathbf{D})) &= \mathbf{B}(\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})) - (\mathbf{A} \cdot \mathbf{B})(\mathbf{C} \times \mathbf{D}). \end{aligned} \quad (1.18)$$

Problem 1.5 Prove the **BAC-CAB** rule by writing out both sides in component form.

Problem 1.6 Prove that

$$[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] = 0.$$

Under what conditions does $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$?

1.1.4 Position, Displacement, and Separation Vectors

The location of a particle in three-dimensional space is specified by its position vector $\mathbf{r}$ relative to a chosen origin O .

1.1.4 Position, Displacement, and Separation Vectors

The location of a point in three dimensions can be described by listing its Cartesian coordinates (x, y, z) . The vector to that point from the origin (Fig. 1.13) is called the **position vector**:

$$\mathbf{r} \equiv x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}}. \quad (1.19)$$

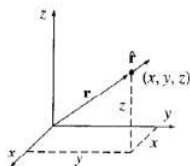


Figure 1.13

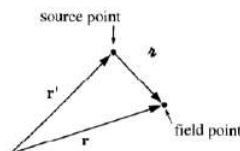


Figure 1.14

I will reserve the letter $\mathbf{r}$ for this purpose, throughout the book. Its magnitude,

$$r = \sqrt{x^2 + y^2 + z^2}, \quad (1.20)$$

is the distance from the origin, and

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} \quad (1.21)$$

is a unit vector pointing radially outward. The **infinitesimal displacement vector**, from (x, y, z) to $(x + dx, y + dy, z + dz)$, is

$$d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}. \quad (1.22)$$

(We could call this $d\mathbf{r}$, since that's what it *is*, but it is useful to reserve a special letter for infinitesimal displacements.)

In electrodynamics one frequently encounters problems involving *two* points—typically, a **source point**, $\mathbf{r}'$, where an electric charge is located, and a **field point**, $\mathbf{r}$, at which you are calculating the electric or magnetic field (Fig. 1.14). It pays to adopt right from the start some short-hand notation for the **separation vector** from the source point to the field point. I shall use for this purpose the script letter $\mathbf{z}$:

$$\mathbf{z} \equiv \mathbf{r} - \mathbf{r}'. \quad (1.23)$$

Its magnitude is

$$z = |\mathbf{r} - \mathbf{r}'|, \quad (1.24)$$

and a unit vector in the direction from $\mathbf{r}'$ to $\mathbf{r}$ is

$$\hat{\mathbf{z}} = \frac{\mathbf{z}}{z} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}. \quad (1.25)$$

In Cartesian coordinates,

$$\mathbf{z} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}, \quad (1.26)$$

$$z = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}, \quad (1.27)$$

$$\hat{\mathbf{z}} = \frac{(x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \quad (1.28)$$

(from which you can begin to appreciate the advantage of the script- $\mathbf{z}$ notation).

Problem 1.7 Find the separation vector $\mathbf{z}$ from the source point $(2, 8, 7)$ to the field point $(4, 6, 8)$. Determine its magnitude (z), and construct the unit vector $\hat{\mathbf{z}}$.

1.1.5 How Vectors Transform

The definition of a vector as “a quantity with a magnitude and direction” is not altogether satisfactory: What precisely does “direction” *mean*?² This may seem a pedantic question, but we shall shortly encounter a species of derivative that *looks* rather like a vector, and we’ll want to know for sure whether it *is* one. You might be inclined to say that a vector is anything that has three components that combine properly under addition. Well, how about this: We have a barrel of fruit that contains N_x pears, N_y apples, and N_z bananas. Is $\mathbf{N} = N_x \hat{\mathbf{x}} + N_y \hat{\mathbf{y}} + N_z \hat{\mathbf{z}}$ a vector? It has three components, and when you add another barrel with M_x pears, M_y apples, and M_z bananas the result is $(N_x + M_x)$ pears, $(N_y + M_y)$ apples, $(N_z + M_z)$ bananas. So it does *add* like a vector. Yet it’s obviously *not* a vector, in

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The answer is that $\mathbf{N}$ *does not transform properly when you change coordinates*. The coordinate frame we use to describe positions in space is of course entirely arbitrary, but there is a specific geometrical transformation law for converting vector components from one frame to another. Suppose, for instance, the $\bar{x}, \bar{y}, \bar{z}$ system is rotated by angle ϕ , relative to x, y, z , about the common $x = \bar{x}$ axis. From Fig. 1.15,

$$A_y = A \cos \theta, \quad A_z = A \sin \theta,$$

while

$$\begin{aligned} \bar{A}_y &= A \cos \bar{\theta} = A \cos(\theta - \phi) = A(\cos \theta \cos \phi + \sin \theta \sin \phi) \\ &= \cos \phi A_y + \sin \phi A_z, \\ \bar{A}_z &= A \sin \bar{\theta} = A \sin(\theta - \phi) = A(\sin \theta \cos \phi - \cos \theta \sin \phi) \\ &= -\sin \phi A_y + \cos \phi A_z. \end{aligned}$$

²This section can be skipped without loss of continuity.

1.1. VECTOR ALGEBRA

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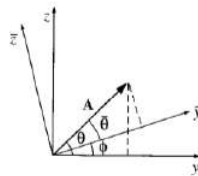


Figure 1.15

We might express this conclusion in matrix notation:

$$\begin{pmatrix} \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}. \quad (1.29)$$

More generally, for rotation about an *arbitrary* axis in three dimensions, the transformation law takes the form

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}, \quad (1.30)$$

or, more compactly,

$$\bar{A}_i = \sum_{j=1}^3 R_{ij} A_j, \quad (1.31)$$

where the index 1 stands for x , 2 for y , and 3 for z . The elements of the matrix R can be ascertained, for a given rotation, by the same sort of geometrical arguments as we used for a rotation about the x axis.

Now: Do the components of $\mathbf{N}$ transform in this way? Of course not—it doesn't matter what coordinates you use to represent positions in space, there is still the same number of apples in the barrel. You can't convert a pear into a banana by choosing a different set of axes, but you *can* turn A_x into $\bar{A}_y$. Formally, then, a *vector is any set of three components that transforms in the same manner as a displacement when you change coordinates*. As always, displacement is the *model* for the behavior of all vectors.

By the way, a (second-rank) **tensor** is a quantity with *nine* components, $T_{xx}, T_{xy}, T_{xz}, T_{yx}, \dots, T_{zz}$, which transforms with *two* factors of R :

$$\begin{aligned} \bar{T}_{xx} &= R_{xx}(R_{xx}T_{xx} + R_{xy}T_{xy} + R_{xz}T_{xz}) \\ &\quad + R_{xy}(R_{xx}T_{yx} + R_{xy}T_{yy} + R_{xz}T_{yz}) \\ &\quad + R_{xz}(R_{xx}T_{zx} + R_{xy}T_{zy} + R_{xz}T_{zz}), \dots \end{aligned}$$

or, more compactly,

$$\bar{T}_{ij} = \sum_{k=1}^3 \sum_{l=1}^3 R_{ik} R_{jl} T_{kl}. \quad (1.32)$$

1.2.1 "Ordinary" Derivatives

Question: Suppose we have a function of one variable: $f(x)$. What does the derivative, df/dx , do for us? **Answer:** It tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, dx :

$$df = \left(\frac{df}{dx} \right) dx. \quad (1.33)$$

In words: If we change x by an amount dx , then f changes by an amount df ; the derivative is the proportionality factor. For example, in Fig. 1.17(a), the function varies slowly with x , and the derivative is correspondingly small. In Fig. 1.17(b), f increases rapidly with x , and the derivative is large, as you move away from $x = 0$.

Geometrical Interpretation: The derivative df/dx is the *slope* of the graph of f versus x .

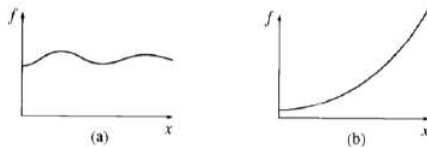


Figure 1.17

1.2.2 Gradient

Suppose, now, that we have a function of *three* variables—say, the temperature $T(x, y, z)$ in a room. (Start out in one corner, and set up a system of axes; then for each point (x, y, z) in the room, T gives the temperature at that spot.) We want to generalize the notion of “derivative” to functions like T , which depend not on *one* but on *three* variables.

Now a derivative is supposed to tell us how fast the function varies, if we move a little distance. But this time the situation is more complicated, because it depends on what *direction* we move: If we go straight up, then the temperature will probably increase fairly rapidly, but if we move horizontally, it may not change much at all. In fact, the question “How fast does T vary?” has an infinite number of answers, one for each direction we might choose to explore.

Fortunately, the problem is not as bad as it looks. A theorem on partial derivatives states that

$$dT = \left(\frac{\partial T}{\partial x} \right) dx + \left(\frac{\partial T}{\partial y} \right) dy + \left(\frac{\partial T}{\partial z} \right) dz. \quad (1.34)$$

This tells us how T changes when we alter all three variables by the infinitesimal amounts dx, dy, dz . Notice that we do *not* require an infinite number of derivatives—*three* will suffice: the *partial* derivatives along each of the three coordinate directions.

Equation 1.34 is reminiscent of a dot product:

$$\begin{aligned} dT &= \left(\frac{\partial T}{\partial x} \hat{\mathbf{x}} + \frac{\partial T}{\partial y} \hat{\mathbf{y}} + \frac{\partial T}{\partial z} \hat{\mathbf{z}} \right) \cdot (dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}) \\ &= (\nabla T) \cdot (d\mathbf{l}), \end{aligned} \quad (1.35)$$

where

$$\nabla T \equiv \frac{\partial T}{\partial x} \hat{\mathbf{x}} + \frac{\partial T}{\partial y} \hat{\mathbf{y}} + \frac{\partial T}{\partial z} \hat{\mathbf{z}} \quad (1.36)$$

is the **gradient** of T . ∇T is a *vector* quantity, with three components; it is the generalized derivative we have been looking for. Equation 1.35 is the three-dimensional version of Eq. 1.33.

Geometrical Interpretation of the Gradient: Like any vector, the gradient has *magnitude* and *direction*. To determine its geometrical meaning, let’s rewrite the dot product (1.35) in abstract form:

$$dT = \nabla T \cdot d\mathbf{l} = |\nabla T| |d\mathbf{l}| \cos \theta, \quad (1.37)$$

where θ is the angle between ∇T and $d\mathbf{l}$. Now, if we fix the *magnitude* $|d\mathbf{l}|$ and search around in various *directions* (that is, vary θ), the *maximum* change in T evidently occurs when $\theta = 0$ (for then $\cos \theta = 1$). That is, for a fixed distance $|d\mathbf{l}|$, dT is greatest when $\mathbf{l}$ move in the *same direction* as ∇T . Thus:

The gradient ∇T points in the direction of maximum increase of the function T .

Moreover:

The magnitude $|\nabla T|$ gives the slope (rate of increase) along this maximal direction.

Imagine you are standing on a hillside. Look all around you, and find the direction of steepest ascent. That is the *direction* of the gradient. Now measure the *slope* in that direction (rise over run). That is the *magnitude* of the gradient. (Here the function we’re talking about is the height of the hill, and the coordinates it depends on are positions—latitude and longitude, say. This function depends on only *two* variables, not *three*, but the geometrical meaning of the gradient is easier to grasp in two dimensions.) Notice from Eq. 1.37 that the direction of maximum *descent* is opposite to the direction of maximum *ascent*, while at right angles ($\theta = 90^\circ$) the slope is zero (the gradient is perpendicular to the contour lines). You can conceive of surfaces that do not have these properties, but they always have “kinks” in them and correspond to nondifferentiable functions.

What would it mean for the gradient to vanish? If $\nabla T = 0$ at (x, y, z) , then $dT = 0$