Functions, Composition of Functions and Invertible Functions

Introduction to Functions

A function is a foundational concept in mathematics that formalizes the notion of a mathematical relationship where each input is associated with exactly one output.

Formally, let A and B be two non-empty sets. A **function** f from A to B (denoted as $f : A \to B$) is a rule of correspondence that assigns to each element $x \in A$ a unique element $y \in B$.

- **Domain:** The set A is called the domain of f.
- **Codomain:** The set B is called the codomain of f.
- **Range (Image):** The subset of B consisting of all actual outputs, defined as $\text{Range}(f) = \{f(x) \mid x \in A\} \subseteq B$.

Classification of Functions

Functions are classified based on how they map elements from the domain to the codomain:

1. **One-to-One (Injective):** A function $f : A \to B$ is injective if distinct elements in the domain map to distinct elements in the codomain.

$$f(x_1) = f(x_2) \implies x_1 = x_2 \quad \text{for all } x_1, x_2 \in A$$

2. **Onto (Surjective):** A function $f : A \to B$ is surjective if every element in the codomain has at least one pre-image in the domain.

$$\text{For every } y \in B, \text{ there exists some } x \in A \text{ such that } f(x) = y$$

3. **Bijective (One-to-One and Onto):** A function is bijective if it is both injective and surjective. Bijectivity is a strict requirement for a function to be invertible.

Composition of Functions

Function composition is an operation that takes two functions and produces a third function by applying one function to the result of another.

Definition

Let $f : A \to B$ and $g : B \to C$ be two functions. The **composition** of g and f, denoted by $g \circ f$ (read as "g circle f"), is a function from A to C defined by:

$$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in A$$

For the composition $g \circ f$ to be valid, the codomain of the inner function (f) must be a subset of the domain of the outer function (g).

Properties of Composition

- **Non-Commutativity:** In general, composition is not commutative. That is, $g \circ f \neq f \circ g$ (even when both compositions are well-defined).
- **Associativity:** If $f : A \to B$, $g : B \to C$, and $h : C \to D$, then composition is associative:

$$(h \circ g) \circ f = h \circ (g \circ f)$$

- **Composition with Identity:** If I_A is the identity function on A ($I_A(x) = x$), then $f \circ I_A = f$ and $I_B \circ f = f$.

Invertible Functions

An invertible function is one that can be "undone." If a function maps an input x to an output y, its inverse maps that output y back to the original input x.

Definition

Let $f : A \to B$ be a function. f is said to be **invertible** if there exists a function $g : B \to A$ such that:

$$g \circ f = I_A \quad \text{and} \quad f \circ g = I_B$$

where I_A and I_B are the identity functions on sets A and B respectively. The function g is uniquely determined by f and is called the **inverse** of f, denoted by f^{-1}.

The Fundamental Criterion for Invertibility

A function $f : A \to B$ is invertible if and only if f is a **bijective** function (both injective and surjective).

- If f is not injective, two different inputs yield the same output, making it impossible to uniquely trace an output back to a single input.
- If f is not surjective, some elements in the codomain have no pre-image, leaving the inverse function undefined for those elements.

Finding the Inverse of a Function

To find the inverse of an invertible function $f(x)$:

1. Replace $f(x)$ with y.
2. Interchange x and y (or solve the equation explicitly for x in terms of y).
3. Express the resulting equation as $f^{-1}(x)$.

Example: Find the inverse of $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x - 5$.

1. Set $y = 3x - 5$.
2. Solve for x: $y + 5 = 3x \implies x = \frac{y+5}{3}$.
3. Replace y with x to write the inverse: $f^{-1}(x) = \frac{x+5}{3}$.

Properties of Inverses

- **Involution:** The inverse of the inverse of a function is the function itself:

$$(f^{-1})^{-1} = f$$

- **Inverse of a Composition:** If $f : A \to B$ and $g : B \to C$ are both invertible functions, then their composition $g \circ f$ is also invertible, and its inverse is given by reversing the order of composition:

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$