Advanced Convergence Tests

Cauchy's Root Test, Raabe's Test, & Logarithmic Test

Subject: Real Analysis

When analyzing infinite series $\sum a_n$ with positive terms ($a_n > 0$), basic tests such as the Comparison Test or D'Alembert's Ratio Test may fail or prove algebraically cumbersome. This lecture covers three higher-level convergence tests designed for these complex scenarios.

1. Cauchy's Root Test

Cauchy's Root Test is particularly effective when the general term a_n involves an exponent containing n.

Mathematical Statement

Let $\sum a_n$ be a series of positive terms, and define the limit:

$$L = \lim_{n \to \infty} (a_n)^{1/n}$$

- If $L < 1$, the series **converges**.
- If $L > 1$, the series **diverges**.
- If $L = 1$, the test is **inconclusive**.

Worked Example 1: Cauchy's Root Test

Problem: Test the convergence of $\sum_{n=1}^{\infty} (n / (2n + 1))^n$.

Solution:

1. Identify $a_n = (n / (2n + 1))^n$.
2. Compute limit L:

$$L = \lim_{n \to \infty} [(n / (2n + 1))^n]^{1/n} = \lim_{n \to \infty} n / (2n + 1)$$

3. Divide numerator and denominator by n:

$$L = \lim_{n \to \infty} 1 / (2 + 1/n) = 1/2$$

Since $L = 1/2 < 1$, the series **converges**.

2. Raabe's Test

Raabe's Test serves as a secondary test when D'Alembert's Ratio Test yields $L = 1$.

Mathematical Statement

Let $\sum a_n$ be a series of positive terms. Compute the limit:

$$L = \lim_{n \to \infty} n \left[(a_n / a_{n+1}) - 1 \right]$$

- If $L > 1$, the series **converges**.
- If $L < 1$, the series **diverges**.
- If $L = 1$, the test fails.

Worked Example 2: Raabe's Test

Problem: Test convergence of $\sum_{n=1}^{\infty} [\, 1 \cdot 3 \cdot 5 \cdots (2n-1) \,] / [\, 2 \cdot 4 \cdot 6 \cdots (2n) \,]$.

Solution:

1. Ratio of successive terms gives $a_n / a_{n+1} = (2n + 2) / (2n + 1)$.
2. Notice that $\lim_{n \to \infty} (a_n / a_{n+1}) = 1$, so Ratio Test fails.
3. Apply Raabe's Test limit:

$$L = \lim_{n \to \infty} n \left[(2n + 2)/(2n + 1) - 1 \right] = \lim_{n \to \infty} n \left[1 / (2n + 1) \right] = 1/2$$

Since $L = 1/2 < 1$, the series **diverges**.

3. Logarithmic Test

The Logarithmic Test is ideal when D'Alembert's test fails and terms contain exponential terms or Euler's constant e.

Mathematical Statement

$$L = \lim_{n \to \infty} \left[n \cdot \ln(a_n / a_{n+1}) \right]$$

- If $L > 1$, the series **converges**.
- If $L < 1$, the series **diverges**.
- If $L = 1$, the test fails.

Summary Strategy for Positive Term Series

1. Does a_n have an overall n-th exponent? → **Cauchy's Root Test**
2. Does D'Alembert Ratio Test give limit equal to 1?
   - If algebraic terms remain → **Raabe's Test**
   - If e or exponential functions remain → **Logarithmic Test**