Double Integrals Lecture Notes

Cartesian & Polar Coordinates with Worked Examples

Subject: Multivariable Calculus

A double integral evaluates the volume under a surface $z = f(x,y)$ over a two-dimensional region R in the xy-plane.

1. Double Integrals in Cartesian Coordinates

$$\iint_R f(x,y)\, dA = \iint_R f(x,y)\, dx\, dy = \iint_R f(x,y)\, dy\, dx$$

Region Types & Order of Integration

- **Type I (Vertically Simple):** $R = \{ (x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x) \}$
  $\iint_R f(x,y)\, dA = \int_a^b [\int_{g1(x)}^{g2(x)} f(x,y)\, dy]\, dx$
- **Type II (Horizontally Simple):** $R = \{ (x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y) \}$
  $\iint_R f(x,y)\, dA = \int_c^d [\int_{h1(y)}^{h2(y)} f(x,y)\, dx]\, dy$

Worked Example 1: Evaluating a Cartesian Double Integral

Problem: Evaluate $\iint_R xy\, dA$, where R is bounded by $y = x^2$ and $y = x$.

Solution:

1. Intersection points: $x^2 = x \Rightarrow x = 0$ and $x = 1$. Limits: $0 \le x \le 1$ and $x^2 \le y \le x$.
2. Set up: $I = \int_0^1 \int_{x^2}^x xy\, dy\, dx$.
3. Inner integral: $\int_{x^2}^x xy\, dy = [x y^2 / 2]_{x^2}^x = (x/2)(x^2 - x^4)$.
4. Outer integral: $I = (1/2) \int_0^1 (x^3 - x^5)\, dx = (1/2) [1/4 - 1/6] = 1/24$.

Worked Example 2: Changing Order of Integration

Problem: Evaluate $\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$.

Solution:

1. Original limits: $y \le x \le 1$ and $0 \le y \le 1$.
2. Re-describe region: $0 \le x \le 1$ and $0 \le y \le x$.
3. Set up & evaluate: $\int_0^1 \int_0^x e^{x^2}\, dy\, dx = \int_0^1 x\, e^{x^2}\, dx$.
4. Using substitution $u = x^2$: $I = [(1/2)\, e^{x^2}]_0^1 = (e - 1) / 2$.

2. Double Integrals in Polar Coordinates

Coordinate Transformations

When regions have circular symmetry or integrands contain $x^2 + y^2$, convert using:

- $x = r \cos \theta$, $y = r \sin \theta$, and $x^2 + y^2 = r^2$
- Area differential: $dA = r\, dr\, d\theta$

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta)\, r\, dr\, d\theta$$

Worked Example 3: Polar Conversion for Circular Region

Problem: Evaluate $\iint_R (x^2 + y^2)\, dA$ over the top half of unit disk $x^2 + y^2 \leq 1$.

Solution:

1. Polar region: $0 \leq \theta \leq \pi$ and $0 \leq r \leq 1$.
2. Set up integral: $I = \int_0^\pi \int_0^1 (r^2)\, r\, dr\, d\theta = \int_0^\pi \int_0^1 r^3\, dr\, d\theta$.
3. Inner integral: $\int_0^1 r^3\, dr = 1/4$.
4. Outer integral: $I = \int_0^\pi (1/4)\, d\theta = \pi/4$.

Worked Example 4: Gaussian Integration

Problem: Evaluate $\iint_R e^{-(x^2+y^2)}\, dA$ over the first quadrant ($x \geq 0, y \geq 0$).

Solution:

1. Polar region: $0 \leq \theta \leq \pi/2$ and $0 \leq r < \infty$.
2. Set up: $I = \int_0^{\pi/2} \int_0^\infty e^{-r^2}\, r\, dr\, d\theta$.
3. Inner integral: $[\, -(1/2)\, e^{-r^2}\,]_0^\infty = 1/2$.
4. Outer integral: $I = (1/2)\, (\pi/2) = \pi/4$.

3. Comparison & Strategy

Feature	Cartesian Coordinates (x,y)	Polar Coordinates (r,θ)
Best Used For	Rectangular, triangular, or parabolic regions	Circles, cardioids, sectors, or rings
Integrand Indicator	Polynomials $x^a y^b$, linear terms	x^2+y^2, $\sqrt{(x^2+y^2)}$, $\arctan(y/x)$
Differential dA	$dx\, dy$ or $dy\, dx$	$r\, dr\, d\theta$ (always include factor r)