UNIT-1
Vector Analysis

Structure of the Unit

1.0 Objectives

Vector analysis is a mathematical shorthand. The vector form helps to provide a clearer understanding of the physical laws. This makes the calculus of the vector functions the natural instrument for the physicist and engineers in solid mechanics, electromagnetism, and so on. To meet objectives ,we emphasize the physical interpretation of vector functions.

1.1 Introduction

Vector algebra is introduced early in the text. The unit deals with vector functions and extends the differential calculus to these vector functions. We finally discuss physical meaning of three important concepts namely ,the gradient, divergence and curl.

1.2 Scalar Product

Scalar Product (dot product) of two vectors $\vec{A}$ and $\vec{B}$ is defined as

$$\boxed{\vec{A}.\vec{B} = AB\cos\theta} \qquad \text{where } 0 \le \theta \le \pi$$

Here θ is the angle between $\vec{A}$ and $\vec{B}$. Note that $\vec{A}.\vec{B}$ is a scalar quantity.

General Properties of Scalar Product:-

(i) $\vec{A}.\vec{B} = \vec{B}.\vec{A}$ (Commutative Law)

(ii) $\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B}+\vec{A}.\vec{C}$

(iii) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

$\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

(iv) If $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

$\vec{A}.\vec{B} = A_1B_1 + A_2B_2 + A_3B_3$

$\vec{A}.\vec{A} = A^2 = A_1^2 + A_2^2 + A_3^2$

(v) $\left|\vec{A}.\vec{B}\right| \le \left|\vec{A}\right|\left|\vec{B}\right|$

(vi) $\vec{A}.\vec{B}$ is independent of co-ordinate system.

Typical Applications of Scalar Product:-

(i) If $\vec{A} \ne 0$, $\vec{B} \ne 0$ and $\vec{A}.\vec{B} = 0$, then $\vec{A}$ and $\vec{B}$ will be perpendicular.

(ii) Component of vector $\vec{A}$ along $\hat{n}$ direction is $\boxed{\vec{A}.\hat{n}}$, where $\hat{n}$ is unit vector.

(iii) Angle between $\vec{A}$ and $\vec{B}$ can be found out by $\boxed{\cos\theta = \dfrac{\vec{A}.\vec{B}}{AB} = \hat{A}.\hat{B}}$

Example1.1 Find the angle between side AC and side AB of a triangle ABC. Coordinates of the vertices A,B,C are $(1+2\sqrt{3},1,2)$, $(1,1,2)$, $(1,3,2)$ respectively.

Sol. $\vec{AB}$ =Position vector of $\vec{B}$ – Position Vector of $\vec{A}$

$$\overrightarrow{AB} = \left(\hat{i}+\hat{j}+2\hat{k}\right) - \left[\left(1+2\sqrt{3}\right)\hat{i}+\hat{j}+2\hat{k}\right] = -2\sqrt{3}\,\hat{i}$$

$$\Rightarrow \quad \left|\vec{AB}\right| = 2\sqrt{3}$$

$$\vec{AC} = \text{P.V. of } \vec{C} \text{ - P.V. of } \vec{A}$$

$$\overrightarrow{AC} = \left(\hat{i}+3\hat{j}+2\hat{k}\right) - \left[\left(1+2\sqrt{3}\right)\hat{i}+\hat{j}+2\hat{k}\right]$$

$$= -2\sqrt{3}\hat{i} + 2\hat{j}$$

$$\Rightarrow \quad \left|\vec{AC}\right| = \sqrt{\left(2\sqrt{3}\right)^2 + 2^2} = 4$$

By dot product

$$\cos\theta = \frac{\vec{AB}.\vec{AC}}{\left|\vec{AB}\right|\left|\vec{AC}\right|} = \frac{\left(-2\sqrt{3}\,\hat{i}\right).\left(-2\sqrt{3}\hat{i}+2\hat{j}\right)}{\left(2\sqrt{3}\right).4} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = 30^0$$

1.3 Vector Product (Cross Product)

Vector product of two vectors $\vec{A}$ and $\vec{B}$ is defined as $\boxed{\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}}$ where $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector in direction of $\left(\vec{A}\times\vec{B}\right)$. Direction of unit victor $\hat{n}$ is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$ and it is given by right handed system.

General Properties of Vector Product:-

(i) If $\vec{A}$ and $\vec{B}$ are parallel or antiparallel i.e. collinear , then $\vec{A}\times\vec{B}=0$

$$\vec{A}\times\vec{A}=0$$

$$\hat{i}\times\hat{i}=0,\ \hat{j}\times\hat{j}=0,\ \hat{k}\times\hat{k}=0$$

(ii) $\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$ (Anti commutative law)

(iii) $\vec{A}\times\left(\vec{B}+\vec{C}\right)=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$

(iv) $\hat{i}\times\hat{j}=\hat{k},\quad \hat{j}\times\hat{k}=\hat{i},\quad \hat{k}\times\hat{i}=\hat{j},$

$$\hat{j}\times\hat{i}=-\hat{k},\quad \hat{k}\times\hat{j}=-\hat{i},\ \hat{i}\times\hat{k}=-\hat{j},$$

(v) If $\vec{A}=A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$ and $\vec{B}=B_1\hat{i}+B_2\hat{j}+B_3\hat{k}$,Vector product of two vectors $\vec{A}$ and $\vec{B}$ as a determinant is represented as given below

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix}$$

$$\vec{A}\times\vec{B}=\hat{i}\left(A_2B_3-A_3B_2\right)-\hat{j}\left(A_1B_3-A_3B_1\right)+\hat{k}\left(A_1B_2-A_2B_1\right)$$

Typical Applications of Vector Product:-

(i) Area of the parallelogram with sides $\vec{A}$ and $\vec{B}$ is $\left|\vec{A}\times\vec{B}\right|$

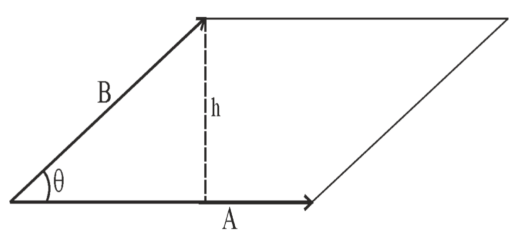

Figure 1.1

Area=Ah=AB $\sin\theta$

$$=\left|\vec{A}\times\vec{B}\right|$$

(ii) Area of the Triangle with sides $\vec{A}$ and $\vec{B}$ is $\frac{1}{2}\left|\vec{A}\times\vec{B}\right|$

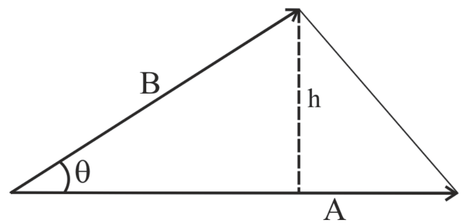

Figure 1.2

Area $= \frac{1}{2}Ah = \frac{1}{2}AB\sin\theta = \frac{1}{2}\left|\vec{A}\times\vec{B}\right|$

(iii) Unit vector that is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$ is given by

$$\boxed{\hat{n} = \pm\frac{\vec{A}\times\vec{B}}{\left|\vec{A}\times\vec{B}\right|}}$$

Example1.2 A solid sphere is rotating with an angular velocity 30 r.p.m. about a fixed axis MN. Position vectors of the points M and N of the sphere are $\left(\hat{i}+2\hat{j}+3\hat{k}\right)m$ and $\left(4\hat{i}+5\hat{j}+6\hat{k}\right)m$ respectively. There is an insect at a point $\left(2\hat{i}-2\hat{j}+5\hat{k}\right)m$ on the surface of the sphere. Calculate the speed of the insect.

Sol. Angular Velocity ω =30 rev. per minute $= 30\times\frac{2\pi}{60}\frac{rad}{\sec} = \pi\frac{rad}{\sec}$

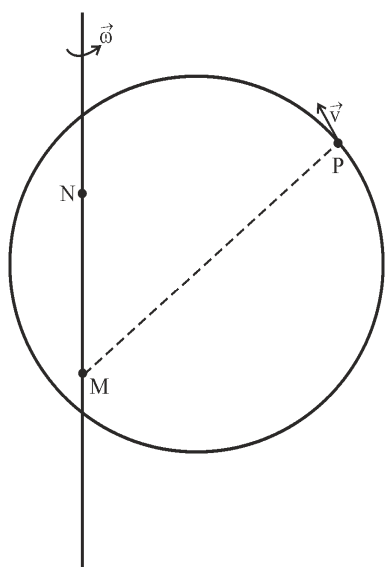

Figure 1.3

Angular velocity $\vec{\omega}$ is an axial vector. Let it be along $\vec{MN}$.

$$\vec{MN} = \vec{N} - \vec{M} = \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$$

$$= 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\frac{\vec{MN}}{\left|\vec{MN}\right|} = \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{\sqrt{3^2 + 3^2 + 3^2}} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$$

$$\vec{\omega} = \omega . \frac{\vec{MN}}{\left|\vec{MN}\right|} = \frac{\pi}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$$

Position vector of the point P with respect to point M is

$$\vec{r} = \vec{P} - \vec{M}$$

$$\vec{r} = \left(2\hat{i} - 2\hat{j} + 5\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) = \hat{i} - 4\hat{j} + 2\hat{k}$$

Linear velocity of P is

$$\vec{v} = \vec{\omega} \times \vec{r} = \frac{\pi}{\sqrt{3}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -4 & 2 \end{vmatrix}$$

$$= \frac{\pi}{\sqrt{3}}\left[\hat{i}\left\{2 - (-4)\right\} - \hat{j}\left(2 - 1\right) + \hat{k}\left(-4 - 1\right)\right]$$

$$= \frac{\pi}{\sqrt{3}}\left[6\hat{i} - \hat{j} - 5\hat{k}\right]$$

$$v = \frac{\pi}{\sqrt{3}}\sqrt{36 + 1 + 25} = \pi\sqrt{\frac{62}{3}}\,\frac{m}{\sec}$$

1.4 Scalar Triple Product $\vec{A}.\left(\vec{B} \times \vec{C}\right)$

$$\boxed{\vec{A}.\left(\vec{B} \times \vec{C}\right) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}}$$

where $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\vec{B} = B_1\hat{i} + B_2\hat{j} + B\hat{k}$$

$$\vec{C} = C_1\hat{i} + C_2\hat{j} + C_3\hat{k}$$

Note that $\vec{A}.\left(\vec{B}\times\vec{C}\right)$ is scalar quantity. We can write scalar triple product $\vec{A}.\left(\vec{B}\times\vec{C}\right)$ as $\left[ABC\right]$ we read $\left[ABC\right]$ as box product.

General Properties of Scalar Triple Product:-

(i) $\vec{A}.\left(\vec{B}\times\vec{C}\right)=\vec{B}.\left(\vec{C}\times\vec{A}\right)=\vec{C}.\left(\vec{A}\times\vec{B}\right)$ i.e. $\left[ABC\right]=\left[BCA\right]=\left[CAB\right]$

(ii) $\left[\hat{i}\,\hat{j}\,\hat{k}\right]=\left[\hat{j}\,\hat{k}\,\hat{i}\right]=\left[\hat{k}\,\hat{i}\,\hat{j}\right]=1$ & $\left[\hat{i}\,\hat{k}\,\hat{j}\right]=-1$

(iii) Magnitude of the scalar triple product $\vec{A}.\left(\vec{B}\times\vec{C}\right)$ of three vectors is equal to the volume of a parallelepiped having sides $\vec{A},\vec{B}$ and $\vec{C}$

(iv) $\vec{A}.\left(\vec{A}\times\vec{C}\right)=0$ *i.e.* $\left[A\,A\,C\right]=0$

$\vec{C}.\left(\vec{A}\times\vec{A}\right)=0$ *i.e.* $\left[C\,A\,A\right]=0$

(v) If scalar triple product vanishes i.e. $\vec{A}.\left(\vec{B}\times\vec{C}\right)=0$ then $\vec{A},\vec{B}$ and $\vec{C}$ are coplanar. In that case volume of parallelepiped formed by them is zero

Note: $\left(\vec{A}.\vec{B}\right)\times\vec{C}$ is meaningless. Similarly $\left(\vec{A}.\vec{B}\right).\vec{C}$ is also meaningless.

Geometrical Interpretation of Scalar Triple Product: -

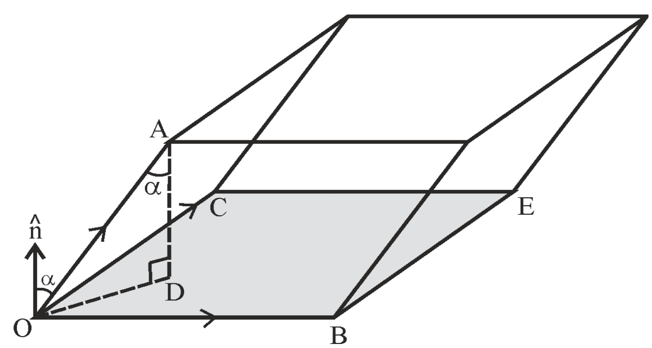

Figure 1.4

$$\vec{OA}=\vec{A}\ ,\ \vec{OB}=\vec{B},\quad \vec{OC}=\vec{C}$$

$$\vec{B}\times\vec{C}=B\ C\sin\theta\ \hat{n}$$

$= (\text{Area of Parallelogram OBEC})\ \hat{n}$

$= S\hat{n}$

Here $\vec{A}.\hat{n} = A\,.\,1.\cos\alpha = AD = height = h$

Thus $\vec{A}.\left(\vec{B}\times\vec{C}\right) = \vec{A}.\left(S\hat{n}\right)$

$= S\,\vec{A}.\,\hat{n} = Sh$

= Volume of parallelepiped

1.5 Vector Triple Product $\vec{A}\times\left(\vec{B}\times\vec{C}\right)$

$$\boxed{\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \vec{B}\left(\vec{A}.\vec{C}\right) - \vec{C}\left(\vec{A}.\vec{B}\right)}$$

Note that $\vec{A}\times\left(\vec{B}\times\vec{C}\right)$ is a vector quantity

General Properties:-

(i) $\boxed{\vec{A}\times\left(\vec{B}\times\vec{C}\right) + \vec{B}\times\left(\vec{C}\times\vec{A}\right) + \vec{C}\times\left(\vec{A}\times\vec{B}\right) = 0}$

(ii) We have in general $\vec{A}\times\left(\vec{B}\times\vec{C}\right) \neq \left(\vec{A}\times\vec{B}\right)\times\vec{C}$

1.6 Gradient $\vec{\nabla}\phi$

Vector differentiable operator $\vec{\nabla}$ is defined as $\boxed{\vec{\nabla} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$

$\vec{\nabla}$ is called 'del'

Gradient: if $\phi(x, y, z)$ is a differentiable scalar field then gradient of ϕ is defined by $\boxed{grad\phi = \vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi}$

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Note that $\vec{\nabla}\phi$ is a vector field .

General Properties:-

(i) $\vec{\nabla}(\phi_1+\phi_2)=\vec{\nabla}\phi_1+\vec{\nabla}\phi_2$

(ii) $\vec{\nabla}(C\phi)=C\vec{\nabla}\phi$ where C is constant

(iii) If ϕ =constant then $\vec{\nabla}\phi=0$

Geometrical Interpretation of gradient:-

"The magnitude of this $\vec{\nabla}\phi$ is equal to maximum value of rate of change of ϕ with distance."

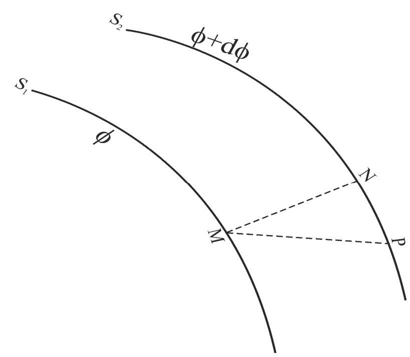

Figure 1.5

Consider a surface S_1 that has constant potential ϕ. At distance MN, there is another surface S_2 which has constant potential $\phi+d\phi$. Here MN is the shortest distance between the two surface s_1 and s_2. If we move from s_1 to s_2, then change in ϕ *is* $d\phi$ .Rate of change of ϕ with distance is highest along the normal MN. So gradϕ at point M is directed along MN.

"$\vec{\nabla}\phi$ points in the direction of maximum rate of increase of the function ϕ with space"

"For any point on the constant surface, direction of vector $\vec{\nabla}\phi$ at that point will be normal to the constant ϕ surface."

Directional Derivate:

The component of gradϕ in the direction of vector $\vec{b}$ is equal to $\vec{\nabla}\phi.\hat{b}$, it is called directional derivate of ϕ in direction of vector $\vec{b}$

Example1.3 Scalar Potential ϕ is given by $\phi = x + y + z$ and an ellipsoid is given by $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3$

Find

(i) $\vec{\nabla}\phi$ at point (1,2,3)

(ii) Unit vector, that is normal to ellipsoid surface at the point (1,2,3)

(iii) Directional derivative of ϕ in the direction of the outward normal of the given ellipsoid at the point (1,2,3)

Sol. (i) $\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x + y + z)$

$$= \hat{i}\frac{\partial}{\partial x}(x + y + z) + \hat{j}\frac{\partial}{\partial y}(x + y + z) + \hat{k}\frac{\partial}{\partial z}(x + y + z)$$

$$= \hat{i} + \hat{j} + \hat{k}$$

which is constant vector and independent of the position of point

(ii) For ellipsoid surface $\psi(x, y, z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9} =$ constant, then $\vec{\nabla}\psi$ will be perpendicular to the surface $\psi(x, y, z)$=constant

$$\vec{\nabla}\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2 + \frac{y^2}{4} + \frac{z^2}{9}\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\left(x^2 + \frac{y^2}{4} + \frac{z^2}{9}\right) + \hat{j}\frac{\partial}{\partial y}\left(x^2 + \frac{y^2}{4} + \frac{z^2}{9}\right) + \hat{k}\frac{\partial}{\partial z}\left(x^2 + \frac{y^2}{4} + \frac{z^2}{9}\right)$$

$$= 2x\hat{i} + \frac{y}{2}\hat{j} + \frac{2z}{9}\hat{k}$$

At the point $(1, 2, 3)$

$$\vec{\nabla}\psi = 2\hat{i} + \hat{j} + \frac{2}{3}\hat{k}$$

$$\left|\vec{\nabla}\psi\right| = \frac{7}{3}$$

Unit vector $\hat{n} = \frac{\vec{\nabla}\psi}{\left|\vec{\nabla}\psi\right|} = \frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}$

Another unit vector normal to the surface is $-\left(\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}\right)$

Its direction is opposite to that above.

(iii) Required directional derivate $\nabla\phi \cdot \hat{n}$

$$= \left(\hat{i} + \hat{j} + \hat{k}\right) \cdot \left(\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}\right)$$

$$= \frac{6 + 3 + 2}{7} = \frac{11}{7}$$

Example1.4 Electric potential due to positive point charge Q is given by $V = \frac{kQ}{r}$ .Find grad V

Sol. $\vec{\nabla}V = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\frac{kQ}{r}$

$$\vec{\nabla}V = kQ\left[\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]$$

$$\vec{\nabla}V = kQ\left[-\frac{\hat{i}}{r^2}\left(\frac{\partial r}{\partial x}\right) - \frac{\hat{j}}{r^2}\left(\frac{\partial r}{\partial y}\right) - \frac{\hat{k}}{r^2}\left(\frac{\partial r}{\partial z}\right)\right]$$

$$\vec{\nabla}V = -\frac{kQ}{r^2}\left[\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right]$$

Here $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r^2 = x^2 + y^2\ z^2$ Partial differentiation with respect to x gives

$$2r\frac{\partial r}{\partial x} = 2x + 0 + 0 \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Thus $\vec{\nabla} V = -\frac{kQ}{r^2}\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right]$

$$= -\frac{kQ}{r^3}\vec{r}$$

$\vec{\nabla} V = -\frac{kQ}{r^3}\vec{r}$ where $\hat{r} = \frac{\vec{r}}{r}$

Physical Interpretation:

Potential V decreases as r increases. Potential $V = \frac{kQ}{r}$ remains same for all points having distance r from the charge Q. Thus equipotential surfaces are spherical type and they are shown with their values of potential.

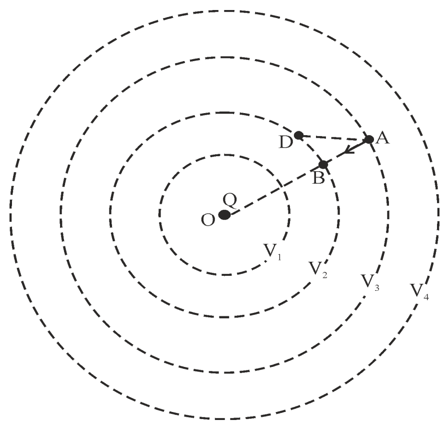

Figure 1.6

Gradient will be in the direction of increasing values of V

Here $V_2 > V_3 > V_4$ so grad$V$ at point A is directed towards V_2 surface. Rate of change of ϕ with distance is highest along the normal AB. Thus at point A, grad$V$ will be in direction of $\vec{AB}$, that is $-\hat{r}$ direction.

Let the charge Q be located at the origin O then

$$\frac{V_2 - V_3}{AB} = \frac{\frac{kQ}{OB} - \frac{kQ}{OB}}{\Delta r} = \frac{\frac{k}{r} - \frac{kQ}{(r+\Delta r)}}{\Delta r} = \frac{kQ}{r(r+\Delta r)}$$

At point A

$\left|\vec{\nabla} V\right|$ =maximum rate of change in V per unit distance

$$= \lim_{\Delta r \to 0}\left(\frac{V_2 - V_3}{AB}\right) = \frac{kQ}{r^2}$$

1.7 Self Learning Exercise-I

Very Short Answer Type Questions

Q.1 Find the angle made by vector $\vec{A} = 4\hat{i} + 4\hat{j} + 3\hat{k}$ with x axis.

Q.2 "If $\vec{A}, \vec{B}, \vec{C}$ are not null vectors and $\vec{A}.\vec{B} = \vec{A}.\vec{C}$ then $\vec{B}$ need not be equal to $\vec{C}$" give an example in favour of above statement.

Short Answer Type Questions

Q.3 If $\vec{a}_1 = \frac{a}{2}\left(-\hat{i} + \hat{j} + \hat{k}\right)$, $\vec{a}_2 = \frac{a}{2}\left(\hat{i} - \hat{j} + \hat{k}\right)$ and $\vec{a}_3 = \frac{a}{2}\left(\hat{i} + \hat{j} - \hat{k}\right)$ represent the primitive translation vectors (sides of primitive cell) of the BCC lattice then find the volume of the primitive cell. Here *a* is the side of conventional cell.

Q.4 Find $\vec{\nabla} r^2$

1.8 Divergence

If $\vec{F}$ is differentiable vector field then divergence of $\vec{F}$ is $\vec{\nabla}.\vec{F}$ which is defined as

$$\vec{\nabla} . \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$\boxed{\vec{\nabla} . \vec{F} = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)}$$

where $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

General Properties of Divergence:-

(i) $\vec{\nabla}.\left(C\vec{A}\right) = C\,\vec{\nabla}.\vec{A}$ where C is constant

(ii) $\vec{\nabla}.\left(\vec{A} + \vec{B}\right) = \vec{\nabla}.\vec{A} + \vec{\nabla}.\vec{B}$

(iii) $\vec{\nabla}.\left(\phi\vec{A}\right) = \vec{\nabla}\phi.\vec{A} + \vec{\nabla}\phi.\vec{A}$

(iv) $\vec{\nabla}.\vec{A} \neq \vec{A}.\vec{\nabla}$

Note that $\vec{A}.\vec{\nabla} = \left(A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}\right)$ is an operator

(v) If $\vec{A}$=constant, then $\vec{\nabla}.\vec{A} = 0$

(vi) $\vec{\nabla}.(\nabla\phi) = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

where $\boxed{\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$ is Laplacian operator

Physical Interpretation of Divergence:

$$\boxed{div\vec{F} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{F}.d\vec{S}}{\Delta V}}$$

Here volume element ΔV is bounded by the infinitesimal surface S in the neighbourhood of a point P. $\oint_S \vec{F}.d\vec{S}$ is the net out flow flux of $\vec{F}$ through infinitesimal surface S.

Thus "divergence of vector field $\vec{F}$ at the point P is equal to net outward flux per unit volume as the volume shrinks to zero in the neighbourhood of the point P."

If div $\vec{F}$ is positive, it means net flux is coming out through infinitesimal volume element at the point P and that point acts as a source.

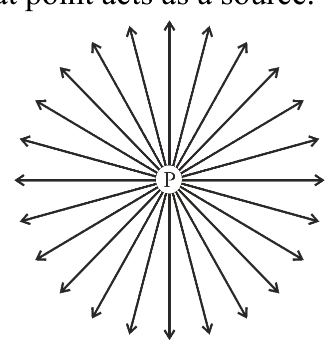

Figure 1.7

In the given figure 1.7, the point P acts as source.

Negative value of div$\vec{F}$ means net flux is going into infinitesimal volume element at the point P and that point acts as a sink.

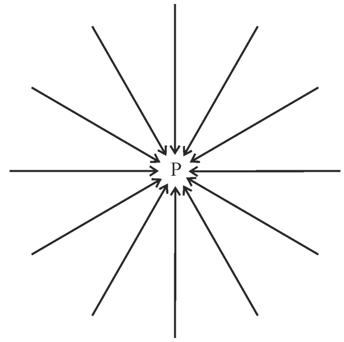

Figure 1.8

If $\vec{\nabla}.\vec{F} = 0$ *then* $\vec{F}$ *is called* ***Solenoidal vector***.

Magnetic field $\vec{B}$ is a solenoid vector $\vec{\nabla}.\vec{B} = 0$ means there is neither source nor sink for field $\vec{B}$. Magnetic field lines always make closed loop. Due to that fact net out flow flux any infinitesimal volume is zero.

Example1.5 Electric field inside a uniformly charged solid sphere is $\vec{E} = \dfrac{\rho \vec{r}}{3 \in_0}$.

Find div$\vec{E}$.

Sol. $\vec{\nabla}.\vec{E} = \vec{\nabla}\left(\dfrac{\rho \vec{r}}{3 \in_0}\right) = \dfrac{\rho}{3 \in_0}\vec{\nabla}.\vec{r}$

$$= \frac{\rho}{3 \in_0}\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{\rho}{3 \in_0}\left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right]$$

$$= \frac{\rho}{3 \in_0}[3]$$

$\vec{\nabla}.\vec{E} = \dfrac{\rho}{\in_0}$ which is Gauss's in electrostatics