Cardinality, Countability & Cantor's Theorem

Lecture Notes on Set Theory, Bijections, and Diagonalization

Subject: Set Theory / Real Analysis

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This lecture covers the fundamental concepts of set cardinality, the distinction between countable and uncountable infinite sets, and Cantor's seminal results on set theory.

1. Cardinality of a Set

The **cardinality** of a set A, denoted by $|A|$ or $card(A)$, measures the size or number of elements in the set. For finite sets, cardinality is simply a non-negative integer. For infinite sets, sizes are compared using bijections.

Equinumerosity (Same Cardinality)

Two sets A and B have the **same cardinality** ($|A| = |B|$) if and only if there exists a **bijection** (a one-to-one and onto function) $f: A \to B$.

Strict Dominance

We write $|A| \leq |B|$ if there exists an **injection** (one-to-one function) $f: A \to B$. If $|A| \leq |B|$ and $|A| \neq |B|$, then $|A| < |B|$.

2. Countable and Uncountable Sets

Definitions

- **Finite Set:** A set that is empty or can be put in bijection with $\{1, 2, 3, ..., n\}$ for some $n \in \mathbb{N}$.
- **Countably Infinite Set:** A set that can be put in bijection with the set of natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$. Its cardinality is denoted by $\aleph_0$ (aleph-null).
- **Countable Set:** A set that is either finite or countably infinite.
- **Uncountable Set:** An infinite set that is not countable (i.e., cannot be put in bijection with $\mathbb{N}$).

Key Properties

1. Any subset of a countable set is countable.
2. The union of countably many countable sets is countable.
3. If A and B are countable, then $A \times B$ is countable. Consequently, $\mathbb{N}^k$ is countable for any $k \in \mathbb{N}$.

Worked Example 1: Proving $\mathbb{Z}$ is Countable

Problem: Show that the set of integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ is countably infinite.

Solution: We construct an explicit bijection $f: \mathbb{N} \to \mathbb{Z}$ that interleaves positive and negative integers:

$$f(n) = n / 2 \text{ if } n \text{ is even} \;\mid\; f(n) = -(n - 1) / 2 \text{ if } n \text{ is odd}$$

Inputs $n = 1, 2, 3, 4, 5, 6, ...$ yield outputs $f(n) = 0, 1, -1, 2, -2, 3, ...$

Since every integer appears exactly once, f is a bijection. Thus, $|\mathbb{Z}| = |\mathbb{N}| = \aleph_0$.

Worked Example 2: Proving ℚ is Countable

Problem: Show that the set of rational numbers ℚ is countable.

Solution:

1. Every rational number can be written uniquely as p/q in lowest terms with $p \in \mathbb{Z}, q \in \mathbb{N}$.
2. Define an injection $g: \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ given by $g(p/q) = (p, q)$.
3. Since ℤ and ℕ are countable, their Cartesian product ℤ × ℕ is countable.
4. Because ℚ maps injectively into a countable set, ℚ is countable.

3. Uncountable Sets & Cantor's Diagonal Argument

Not all infinite sets are the same size. The set of real numbers ℝ is strictly larger than ℕ.

Worked Example 3: Uncountability of the Interval (0,1)

Theorem: The real interval $(0, 1)$ is uncountable.

Proof (Cantor's Diagonalization): Assume by contradiction that $(0,1)$ is countable. List all numbers as a sequence:

$$x_1 = 0.\, d_{11}\, d_{12}\, d_{13}\, d_{14} \ldots$$
$$x_2 = 0.\, d_{21}\, d_{22}\, d_{23}\, d_{24} \ldots$$
$$x_3 = 0.\, d_{31}\, d_{32}\, d_{33}\, d_{34} \ldots$$

Construct a new number $y = 0.\, y_1\, y_2\, y_3 \ldots \in (0,1)$ defining digits: $y_n = 4$ if $d_{nn} \neq 4$, and $y_n = 5$ if $d_{nn} = 4$.

The number y differs from x_n in the n-th decimal place for all n. Thus, y cannot be in our list, contradicting completeness. Hence, $(0,1)$ is uncountable.

4. Cantor's Theorem

Cantor's Theorem: For any set A, $|A| < |$*&P;(A)*$|$.

Proof of Cantor's Theorem

1. **Injection:** Define $f: A \to$ *&P;(A)* by $f(x) = \{x\}$. This is injective, so $|A| \leq |$*&P;(A)*$|$.
2. **Non-existence of Surjection:** Assume there exists a surjection $g: A \to$ *&P;(A)*.
   Define the set $D = \{ x \in A \mid x \notin g(x) \}$.
   Since $D \in$ *&P;(A)* and g is surjective, there exists $d \in A$ such that $g(d) = D$.
   If $d \in D$, then $d \notin g(d) = D$ (contradiction).
   If $d \notin D$, then $d \notin g(d) \Rightarrow d \in D$ (contradiction).

Thus, no surjection exists, proving $|A| < |$*&P;(A)*$|$.

5. Summary Table

Set	Symbol	Description	Cardinality
Natural Numbers	ℕ	{1, 2, 3, ...}	$\aleph_0$ (Countable)
Integers	ℤ	{..., -1, 0, 1, ...}	$\aleph_0$ (Countable)
Rationals	ℚ	{p/q | p ∈ ℤ, q ∈ ℕ}	$\aleph_0$ (Countable)
Reals	ℝ	Real number line	$c = 2^{\aleph_0}$ (Uncountable)
Interval (0,1)	(0,1)	Continuum	$c = 2^{\aleph_0}$ (Uncountable)
Power Set of ℕ	***&P;(ℕ)***	Set of all subsets of ℕ	$2^{\aleph_0}$ (Uncountable)