Area & Surface Area Lecture Notes

Applications of Double Integrals in Cartesian and Polar Coordinates

Subject: Multivariable Calculus

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This document covers methods for computing two-dimensional plane area and three-dimensional surface area using double integrals.

1. Area of a Plane Region

The area A of a 2D region R in the xy-plane is calculated by integrating $f(x,y) = 1$ over R:

$$A = \iint_R dA$$

Cartesian Coordinates

- **Vertically Simple Region (Type I):** $a \le x \le b,\ g_1(x) \le y \le g_2(x)$
  $A = \int_a^b \int_{g1(x)}^{g2(x)} dy\, dx = \int_a^b [\, g_2(x) - g_1(x)\,]\, dx$
- **Horizontally Simple Region (Type II):** $c \le y \le d,\ h_1(y) \le x \le h_2(y)$
  $A = \int_c^d \int_{h1(y)}^{h2(y)} dx\, dy = \int_c^d [\, h_2(y) - h_1(y)\,]\, dy$

Polar Coordinates

When enclosed by polar curves $\alpha \le \theta \le \beta$ and $r_1(\theta) \le r \le r_2(\theta)$:

$$A = \iint_R r\, dr\, d\theta = \int_\alpha^\beta \int_{r1(\theta)}^{r2(\theta)} r\, dr\, d\theta$$

Worked Example 1: Area Bounded by Parabola and Line

Problem: Find the area enclosed by $y = x^2$ and $y = 2x + 3$.

Solution:

1. Intersection points: $x^2 = 2x + 3 \Rightarrow (x - 3)(x + 1) = 0 \Rightarrow x = -1, x = 3$.
2. Set up integral: $A = \int_{-1}^3 \int_{x^2}^{2x+3} dy\, dx$.
3. Evaluate: $A = \int_{-1}^3 (2x + 3 - x^2)\, dx = [\, x^2 + 3x - x^3/3\,]_{-1}^3 = 32 / 3$.

Worked Example 2: Area of One Loop of a Rose Curve (Polar)

Problem: Find the area of one loop of $r = \cos(2\theta)$.

Solution:

1. Loop bounds: $r = 0 \Rightarrow \cos(2\theta) = 0 \Rightarrow -\pi/4 \le \theta \le \pi/4$.
2. Set up integral: $A = \int_{-\pi/4}^{\pi/4} \int_0^{\cos(2\theta)} r\, dr\, d\theta$.
3. Inner integral: $\int_0^{\cos(2\theta)} r\, dr = (1/2) \cos^2(2\theta)$.
4. Outer integral: $A = (1/4) \int_{-\pi/4}^{\pi/4} (1 + \cos(4\theta))\, d\theta = \pi / 8$.

2. Surface Area of a 3D Surface

The surface area S of $z = f(x,y)$ projected over region R in the xy-plane is:

$$S = \iint_R dS = \iint_R \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\, dA$$

Worked Example 3: Surface Area of a Paraboloid Portion

Problem: Find the surface area of $z = x^2 + y^2$ below $z = 4$.

Solution:

1. Region R: $x^2 + y^2 \le 4$ (disk of radius 2: $0 \le \theta \le 2\pi$, $0 \le r \le 2$).
2. Partial derivatives: $\partial z/\partial x = 2x$, $\partial z/\partial y = 2y$.
3. Surface element: $dS = \sqrt{1 + 4x^2 + 4y^2}\, dA = \sqrt{1 + 4r^2}\, r\, dr\, d\theta$.
4. Integral evaluation: $S = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = (\pi / 6)(17\sqrt{17} - 1)$.

3. Key Formulas Reference

Concept	Coordinate System	Formula
Plane Area	Cartesian	$A = \iint_R dx\, dy$
Plane Area	Polar	$A = \iint_R r\, dr\, d\theta$
Surface Area (z = f(x,y))	Cartesian	$S = \iint_R \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy$
Surface Area (z = f(r,θ))	Polar	$S = \iint_R \sqrt{1 + z_r^2 + (1/r^2) z_\theta^2}\, r\, dr\, d\theta$