Linear Algebra: Lecture Notes on Matrices

Definitions, Classifications, Structural Types & Algebraic Operations

Subject: Linear Algebra

This document provides a detailed overview of matrix theory, covering standard notations, structural classifications (kinds of matrices), algebraic operations, operational properties, and a fully worked numerical example.

1. Introduction to Matrices

A **matrix** is a rectangular array of numbers, symbols, or expressions arranged in horizontal rows and vertical columns. The individual values inside a matrix are called its *elements* or *entries*.

Standard Notation & Dimensions

A matrix A with m rows and n columns is said to have an **order** (or dimension) of $m \times n$:

$$A = [a_{ij}]_{m \times n}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \text{\&ddots;} & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- i denotes the row index ($1 \le i \le m$).
- j denotes the column index ($1 \le j \le n$).
- a_{ij} represents the element positioned at the i-th row and j-th column.

2. Kinds (Types) of Matrices

A. Dimensional Types

- **Row Matrix:** A matrix having only one row ($1 \times n$). → $\begin{bmatrix} 1 & -3 & 5 \end{bmatrix}$
- **Column Matrix:** A matrix having only one column ($m \times 1$). → $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$
- **Square Matrix:** A matrix where the number of rows equals the number of columns ($m = n$).
- **Rectangular Matrix:** A matrix where the number of rows does not equal the number of columns ($m \neq n$).

B. Square Matrix Variants

Diagonal Matrix
A square matrix where all off-diagonal elements are zero ($a_{ij} = 0$ for $i \neq j$).

Scalar Matrix
A diagonal matrix where all diagonal entries are equal to a constant scalar k ($a_{ii} = k$).

Identity (Unit) Matrix (I)
A diagonal matrix where all main diagonal elements equal 1 ($a_{ii} = 1$).

Triangular Matrices
Upper: $a_{ij} = 0$ for $i > j$.
Lower: $a_{ij} = 0$ for $i < j$.

C. Special Structural Types

- **Zero (Null) Matrix (O):** A matrix in which every entry is zero ($a_{ij} = 0$ for all i, j).
- **Symmetric Matrix:** A square matrix equal to its transpose ($A = A^T$, meaning $a_{ij} = a_{ji}$).
- **Skew-Symmetric Matrix:** A square matrix equal to its negative transpose ($A = -A^T$, meaning $a_{ij} = -a_{ji}$). All main diagonal entries are zero.
- **Orthogonal Matrix:** A real square matrix whose transpose is equal to its inverse ($A^TA = AA^T = I$).

3. Operations on Matrices

1. Matrix Addition and Subtraction

Two matrices can be added or subtracted **if and only if** they have the exact same order ($m \times n$).

$$C = A \pm B = [a_{ij} \pm b_{ij}]$$

Properties:

- Commutative: $A + B = B + A$
- Associative: $(A + B) + C = A + (B + C)$
- Additive Identity: $A + O = A$

2. Scalar Multiplication

Multiplying a matrix A by a scalar k scales every individual element by k.

$$k\,A = [k \cdot a_{ij}]$$

Properties: $k(A + B) = kA + kB$ | $(k + m)A = kA + mA$

3. Matrix Multiplication

Two matrices A and B can be multiplied to form AB **only if** the number of columns in A equals the number of rows in B.

- **Dimension Rule:** If A is $m \times n$ and B is $n \times p$, the result $C = AB$ is $m \times p$.
- **Product Formula:**

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

• **Properties:** Non-commutative in general ($AB \neq BA$), Associative $(AB)C = A(BC)$, Distributive $A(B + C) = AB + AC$.

4. Transpose of a Matrix (A^T)

Obtained by interchanging the rows and columns of matrix A. If A is $m \times n$, then A^T is $n \times m$.

Reversal Law for Transposition: $(AB)^T = B^T A^T$

Worked Example: Linear Combination & Matrix Product

Problem: Given $A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ and $B = \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix}$, compute $2A - B$ and AB.

1. Calculate $2A - B$:

$2A = 2$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 6 & 8 \end{vmatrix}$$

$2A - B =$

$$\begin{vmatrix} 2 & 4 \\ 6 & 8 \end{vmatrix} - \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} = \begin{vmatrix} 2\text{-}2 & 4\text{-}0 \\ 6\text{-}1 & 8\text{-}3 \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ 5 & 5 \end{vmatrix}$$

2. Calculate AB:

$AB =$

$$\begin{vmatrix} (1\cdot 2 + 2\cdot 1) & (1\cdot 0 + 2\cdot 3) \\ (3\cdot 2 + 4\cdot 1) & (3\cdot 0 + 4\cdot 3) \end{vmatrix} = \begin{vmatrix} 2+2 & 0+6 \\ 6+4 & 0+12 \end{vmatrix} = \begin{vmatrix} 4 & 6 \\ 10 & 12 \end{vmatrix}$$

4. Operations Summary Table

Operation / Property	Required Condition	Key Formula / Rule
Addition / Subtraction	Identical dimensions ($m \times n$)	$c_{ij} = a_{ij} \pm b_{ij}$
Scalar Multiplication	Any matrix	$(kA)_{ij} = k \cdot a_{ij}$
Matrix Multiplication	Cols of A = Rows of B ($n = n$)	$c_{ij} = \sum a_{ik}b_{kj}$
Transpose (A^T)	Any matrix	Swap rows and columns ($a_{ij} \to a_{ji}$)
Symmetric Check	Square matrix ($m = n$)	$A = A^T$
Skew-Symmetric Check	Square matrix ($m = n$)	$A = -A^T$ (Diagonal entries = 0)