ORTHOGONAL TRAJECTORIES

1. Introduction

In many applications of differential equations, we obtain a family of curves depending on an arbitrary constant. Sometimes we want to find another family of curves that intersects every member of the given family at right angles. Such curves are called orthogonal trajectories.

Definition

If two families of curves intersect each other at an angle of 90° at every point of intersection, then each family is called the orthogonal trajectory of the other.

For example, the family $y = cx$ consists of straight lines passing through the origin. Its orthogonal trajectories are circles centered at the origin, $x^2 + y^2 = C$.

2. Geometrical Meaning

Suppose two curves intersect at a point $P(x, y)$. If their slopes at P are m_1 and m_2, then the curves are perpendicular when

$$m_1 m_2 = -1$$

Hence, if the slope of the given family is m, the slope of its orthogonal trajectory is

$$m_2 = -1/m$$

3. General Method

Suppose the given family is $F(x, y, c) = 0$, where c is an arbitrary constant.

1. Differentiate the equation with respect to x.

2. Use the original equation to eliminate the arbitrary constant c.

3. Obtain a differential equation of the form $dy/dx = f(x, y)$.

4. For orthogonal trajectories, replace the slope by $-1/f(x, y)$.

5. Solve the resulting differential equation. The solution is the required family.

4. Fundamental Formula

Given family: $dy/dx = f(x,y)$

Orthogonal trajectories: $dy/dx = -1/f(x,y)$

5. Why is the Product of Slopes −1?

Let the tangent to one curve make an angle ϑ with the positive x-axis. Its slope is $m_1 = \tan\vartheta$. A perpendicular tangent makes an angle $\vartheta + \pi/2$, so $m_2 = \tan(\vartheta + \pi/2) = -\cot\vartheta = -1/\tan\vartheta$. Therefore $m_1 m_2 = -1$.

6. Example 1: Straight Lines Through the Origin

Given $y = cx$.

Differentiating, $dy/dx = c$. From the original equation, $c = y/x$. Hence $dy/dx = y/x$.

For orthogonal trajectories, $dy/dx = -x/y$. Therefore $y\,dy = -x\,dx$. Integrating gives $y^2/2 = -x^2/2 + C$, so $x^2 + y^2 = C$.

Thus the orthogonal trajectories are circles centered at the origin.

7. Example 2: Family of Circles

Given $x^2 + y^2 = 2cy$.

Differentiating: $2x + 2y(dy/dx) = 2c(dy/dx)$. Since $c = (x^2 + y^2)/(2y)$, elimination gives

$dy/dx = 2xy/(x^2 - y^2)$.

Therefore the orthogonal equation is $dy/dx = (y^2 - x^2)/(2xy)$.

Put $y = vx$, so $dy/dx = v + x\,dv/dx$. Then

$x\,dv/dx = -(v^2 + 1)/(2v)$.

Thus $2v/(v^2+1)\,dv = -dx/x$. Integration gives $\ln(v^2+1) = -\ln|x| + C$. Since $v = y/x$, the final result is

$x^2 + y^2 = Cx$.

8. Example 3: Parabolas

Given $y^2 = 4ax$.

Differentiating: $2y(dy/dx) = 4a$. Since $a = y^2/(4x)$, we obtain $dy/dx = y/(2x)$.

Hence the orthogonal equation is $dy/dx = -2x/y$. Thus $y\,dy = -2x\,dx$. On integration,

$y^2/2 = -x^2 + C$,

so

$2x^2 + y^2 = C$.

9. Example 4: Exponential Family

Given $y = ce^x$.

Differentiating gives $dy/dx = ce^x = y$. Therefore the orthogonal equation is $dy/dx = -1/y$.

Thus $y\,dy = -dx$. Integrating,

$y^2/2 = -x + C$,

so

$y^2 + 2x = C$.

10. Example 5: Family $y = cx^2$

Given $y = cx^2$. Differentiation gives $dy/dx = 2cx$. Since $c = y/x^2$, $dy/dx = 2y/x$.

The orthogonal equation is $dy/dx = -x/(2y)$. Hence $2y\,dy = -x\,dx$. Integration gives $y^2 = -x^2/4 + C$, or $x^2 + 4y^2 = C$.

11. Orthogonal Trajectories in Polar Coordinates

For a family $F(r, \vartheta, c) = 0$, the angle ψ between the tangent and the radius vector satisfies $\tan\psi = r(d\vartheta/dr)$. If the two curves intersect orthogonally, then $\tan\psi_1 \tan\psi_2 = -1$. Thus, if the given family satisfies $r(d\vartheta/dr) = f(r,\vartheta)$, its orthogonal family satisfies

$$r\,d\theta/dr = -1/f(r,\theta)$$

12. Example in Polar Coordinates

Find the orthogonal trajectories of $r = c(1 + \cos\vartheta)$. Differentiating with respect to ϑ gives $dr/d\vartheta = -c \sin\vartheta$. Since $c = r/(1+\cos\vartheta)$, we obtain

$$(1/r)(dr/d\theta) = -\sin\theta/(1+\cos\theta)$$

Hence $r(d\vartheta/dr) = -(1+\cos\vartheta)/\sin\vartheta$. For the orthogonal family, $r(d\vartheta/dr) = \sin\vartheta/(1+\cos\vartheta)$. Therefore

$$dr/r = [(1+\cos\theta)/\sin\theta]\,d\theta$$

Using $(1+\cos\vartheta)/\sin\vartheta = \cot(\vartheta/2)$, integration gives $\ln r = 2\ln|\sin(\vartheta/2)| + C$. Hence

$$r = C\sin^2(\theta/2) = C(1-\cos\theta)$$

13. Orthogonal vs. Oblique Trajectories

Orthogonal trajectories intersect at 90°, so $m_1 m_2 = -1$. Oblique trajectories intersect at a constant angle $\alpha \neq 90°$. For slopes m_1 and m_2, the angle formula is

$$\tan\alpha = (m_2 - m_1)/(1 + m_1 m_2)$$

For $\alpha = 90°$, the denominator must vanish, giving $1 + m_1 m_2 = 0$ and therefore $m_1 m_2 = -1$.

14. Important Special Cases

Case 1: Given slope is zero

If $dy/dx = 0$, the given curves are horizontal. Their orthogonal trajectories are vertical curves $x = C$. The reciprocal formula should not be applied directly to a zero slope.

Case 2: Given curve is vertical

A vertical curve has infinite slope. Its orthogonal trajectories are horizontal curves $y = C$.

15. General Worked Example

Find the orthogonal trajectories of $x^2 + y^2 = 2cx$.

Differentiating gives $2x + 2y(dy/dx) = 2c$, so $x + y(dy/dx) = c$. From the original equation $c = (x^2+y^2)/(2x)$. Hence

$$dy/dx = (y^2 - x^2)/(2xy)$$

For orthogonal trajectories:

$$dy/dx = 2xy/(x^2 - y^2)$$

This is homogeneous. Put $y = vx$. Then $dy/dx = v + x\, dv/dx$, and therefore

$$x\, dv/dx = v(1+v^2)/(1-v^2)$$

Thus $(1-v^2)/[v(1+v^2)]\, dv = dx/x$. Since this integrand equals $1/v - 2v/(1+v^2)$, integration gives $\ln|v| - \ln(1+v^2) = \ln|x| + C$. Substituting $v = y/x$ and simplifying yields

$$y/(x^2+y^2) = C$$

or equivalently

$$x^2 + y^2 = Cy$$

16. Geometrical Interpretation

A family of curves can be viewed as a collection of flow lines. At every point on a curve, the tangent gives the direction of the curve. An orthogonal trajectory has a tangent perpendicular to this direction. Thus, if the original family represents $dy/dx = f(x,y)$, the orthogonal family represents $dy/dx = -1/f(x,y)$.

17. Short Algorithm for Examinations

1. Differentiate the given family.

2. Eliminate the arbitrary constant.

3. Obtain $dy/dx = f(x,y)$.

4. Replace the slope by $-1/f(x,y)$.

5. Solve the resulting differential equation.

6. The resulting family is the orthogonal trajectory.

18. Common Mistakes

- *Forgetting the negative sign: the correct relation is $m_2 = -1/m_1$.*
- *Not eliminating the arbitrary constant before forming the differential equation.*
- *Applying the reciprocal formula directly when the given slope is zero.*
- *Not checking the final result. Differentiate the final family and verify that $m_1 m_2 = -1$.*

19. Practice Problems

1. Find the orthogonal trajectories of $y = cx^3$.
Answer: $3y^2 + x^6 = C$.

2. Find the orthogonal trajectories of $y^2 = cx$.
Answer: $y^2 + 4x^2 = C$.

3. Find the orthogonal trajectories of $x^2 + y^2 = c$.
Answer: $y = C$ locally (the orthogonal family consists of radial lines).

4. Find the orthogonal trajectories of $y = ce^{(-x)}$.
Answer: $y^2 - 2x = C$.

5. Find the orthogonal trajectories of $y = cx^n$.
Answer: $ny^2 + x^2 = C$.

20. Summary

The central idea is that two curves intersect orthogonally when the product of their slopes is -1. If the given family has slope $m = dy/dx$, then the orthogonal trajectory has slope $-1/m$. For a family $F(x,y,c)=0$, differentiate, eliminate the arbitrary constant, replace the slope by its negative reciprocal, and solve the resulting differential equation.

$$m_1 m_2 = -1$$

$$dy/dx = f(x,y) \Rightarrow dy/dx = -1/f(x,y)$$