Transpose, Symmetric and Skew-Symmetric Matrices

1. Introduction

Matrices are rectangular arrangements of numbers or other mathematical quantities in rows and columns. In linear algebra, the **transpose**, **symmetric matrix**, and **skew-symmetric matrix** are important concepts with many applications.

Let

$$A = [a_{ij}]$$

be an $m \times n$ matrix.

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2. Transpose of a Matrix

Definition

The **transpose** of a matrix A is obtained by changing its rows into columns and its columns into rows.

The transpose of A is denoted by

$$\boxed{A^T}$$

or sometimes by A'.

If

$$A = [a_{ij}]_{m\times n},$$

then

$$\boxed{A^T = [a_{ji}]_{n\times m}.}$$

Thus, the element in the i-th row and j-th column of A becomes the element in the j-th row and i-th column of A^T.

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3. Example of Transpose

Consider

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

The matrix A has two rows and three columns.

Therefore,

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

Hence,

$$\boxed{A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.}$$

Notice that a 2×3 matrix becomes a 3×2 matrix after transposition.

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4. Transpose of a Square Matrix

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$\boxed{A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.}$$

For a 3×3 matrix,

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

we have

$$\boxed{A^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}.}$$

5. Properties of Transpose

Let A and B be matrices of appropriate orders and let k be a scalar.

Property 1: Transpose of the transpose

$$\boxed{(A^T)^T = A}$$

Property 2: Transpose of a sum

$$\boxed{(A+B)^T = A^T + B^T}$$

Property 3: Transpose of a scalar multiple

$$\boxed{(kA)^T = kA^T}$$

Property 4: Transpose of a difference

$$\boxed{(A-B)^T = A^T - B^T}$$

Property 5: Transpose of a product

If A and B are conformable for multiplication, then

$$\boxed{(AB)^T = B^T A^T.}$$

Notice that **the order is reversed**.

For three matrices,

$$\boxed{(ABC)^T = C^T B^T A^T.}$$

Property 6: Transpose of the identity matrix

$$\boxed{I^T = I}$$

6. Symmetric Matrix

Definition

A square matrix A is called a **symmetric matrix** if

$$\boxed{A^T = A.}$$

In terms of its elements,

$$\boxed{a_{ij} = a_{ji}}$$

for all i, j.

Therefore, the elements on opposite sides of the principal diagonal are equal.

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7. Example of a Symmetric Matrix

Consider

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 5 & 6 \\ 4 & 6 & 7 \end{pmatrix}.$$

Its transpose is

$$A^T = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 5 & 6 \\ 4 & 6 & 7 \end{pmatrix}.$$

Thus,

$$A^T = A.$$

Therefore,

$$\boxed{A \text{ is symmetric.}}$$

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8. General Form of a Symmetric Matrix

A general 2×2 symmetric matrix has the form

$$\boxed{A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.}$$

A general 3×3 symmetric matrix has the form

$$\boxed{A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}.}$$

The entries satisfy

$$a_{ij} = a_{ji}.$$

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9. Properties of Symmetric Matrices

Property 1

A symmetric matrix must be a **square matrix**.

Property 2

If A is symmetric, then

$$\boxed{A^T = A.}$$

Property 3

The diagonal elements of a symmetric matrix can be arbitrary.

Property 4

The entries above and below the principal diagonal are mirror images.

Property 5

If A and B are symmetric matrices of the same order, then

$$A + B$$

is also symmetric.

Indeed,

$$(A + B)^T = A^T + B^T = A + B.$$

Property 6

If A is symmetric, then for any scalar k,

$$\boxed{kA \text{ is symmetric.}}$$

Property 7

If A is symmetric and A^{-1} exists, then

$$\boxed{A^{-1} \text{ is symmetric.}}$$

Because

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}.$$

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10. Skew-Symmetric Matrix

Definition

A square matrix A is called a **skew-symmetric matrix** if

$$\boxed{A^T = -A.}$$

Equivalently,

$$\boxed{a_{ij} = -a_{ji}.}$$

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11. Example of a Skew-Symmetric Matrix

Consider

$$A = \begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{pmatrix}.$$

Then

$$A^T = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{pmatrix}.$$

On the other hand,

$$-A = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{pmatrix}.$$

Therefore,

$$A^T = -A.$$

Hence,

$$\boxed{A \text{ is skew-symmetric.}}$$

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12. Diagonal Elements of a Skew-Symmetric Matrix

This is a very important property.

For a skew-symmetric matrix,

$$A^T = -A.$$

Comparing the i, i-th elements,

$$a_{ii} = -a_{ii}.$$

Therefore,

$$2a_{ii} = 0.$$

Hence,

$$\boxed{a_{ii} = 0.}$$

Thus, **all diagonal elements of a skew-symmetric matrix are zero.**

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13. General Form of a 3×3 Skew-Symmetric Matrix

A general 3×3 skew-symmetric matrix is

$$\boxed{A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.}$$

Notice that

$$a_{12} = -a_{21},$$

$$a_{13} = -a_{31},$$

and

$$a_{23} = -a_{32}.$$

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14. Properties of Skew-Symmetric Matrices

Property 1

A skew-symmetric matrix must be square.

Property 2

All diagonal elements are zero:

$$\boxed{a_{ii} = 0.}$$

Property 3

For $A^T = -A$,

$$\boxed{A^T = -A.}$$

Property 4

If A and B are skew-symmetric matrices of the same order, then

$$A + B$$

is also skew-symmetric.

Indeed,

$$(A+B)^T = A^T + B^T = -A - B = -(A+B).$$

Property 5

If A is skew-symmetric and k is a scalar, then

$$\boxed{kA \text{ is skew-symmetric.}}$$

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15. Difference Between Symmetric and Skew-Symmetric Matrices

Symmetric Matrix	Skew-Symmetric Matrix
$A^T = A$	$A^T = -A$
$a_{ij} = a_{ji}$	$a_{ij} = -a_{ji}$
Square matrix	Square matrix
Diagonal elements can be any numbers	All diagonal elements are zero
Example: $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$	Example: $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

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16. Decomposition of a Square Matrix

One of the most important results is that **every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.**

Let A be any square matrix.

We can write

$$\boxed{A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T).}$$

Define

$$\boxed{S = \frac{1}{2}(A + A^T)}$$

and

$$\boxed{K = \frac{1}{2}(A - A^T).}$$

Then

$$A = S + K.$$

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17. Proof that S is Symmetric

We have

$$S = \frac{1}{2}(A + A^T).$$

Taking transpose,

$$\begin{aligned} S^T &= \frac{1}{2}(A + A^T)^T \\ &= \frac{1}{2}(A^T + (A^T)^T) \\ &= \frac{1}{2}(A^T + A) \\ &= S. \end{aligned}$$

Therefore,

$$\boxed{S \text{ is symmetric.}}$$

18. Proof that K is Skew-Symmetric

We have

$$K = \frac{1}{2}(A - A^T).$$

Taking transpose,

$$\begin{aligned} K^T &= \frac{1}{2}(A - A^T)^T \\ &= \frac{1}{2}(A^T - A) \\ &= -\frac{1}{2}(A - A^T) \\ &= -K. \end{aligned}$$

Therefore,

$$\boxed{K \text{ is skew-symmetric.}}$$

Hence,

$$\boxed{A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew-symmetric}}.}$$

19. Example of Matrix Decomposition

Let

$$A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Then

$$A^T = \begin{pmatrix} 2 & 3 & 7 \\ 4 & 5 & 8 \\ 1 & 6 & 9 \end{pmatrix}.$$

The symmetric part is

$$S = \frac{1}{2}(A + A^T).$$

Therefore,

$$S = \frac{1}{2}\begin{pmatrix} 4 & 7 & 8 \\ 7 & 10 & 14 \\ 8 & 14 & 18 \end{pmatrix},$$

so

$$\boxed{S = \begin{pmatrix} 2 & \frac{7}{2} & 4 \\ \frac{7}{2} & 5 & 7 \\ 4 & 7 & 9 \end{pmatrix}.}$$

The skew-symmetric part is

$$K = \frac{1}{2}(A - A^T).$$

Thus,

$$K = \frac{1}{2}\begin{pmatrix} 0 & 1 & -6 \\ -1 & 0 & -2 \\ 6 & 2 & 0 \end{pmatrix}.$$

Hence,

$$\boxed{K = \begin{pmatrix} 0 & \frac{1}{2} & -3 \\ -\frac{1}{2} & 0 & -1 \\ 3 & 1 & 0 \end{pmatrix}.}$$

Finally,

$$\boxed{A = S + K.}$$

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20. Uniqueness of the Decomposition

Suppose

$$A = S_1 + K_1 = S_2 + K_2,$$

where S_1, S_2 are symmetric and K_1, K_2 are skew-symmetric.

Then

$$S_1 - S_2 = K_2 - K_1.$$

The left-hand side is symmetric, while the right-hand side is skew-symmetric.

A matrix that is both symmetric and skew-symmetric must satisfy

$$A^T = A$$

and

$$A^T = -A.$$

Therefore,

$$A = -A,$$

so

$$2A = 0,$$

and hence

$$A = 0.$$

Therefore,

$$S_1 = S_2, \qquad K_1 = K_2.$$

Thus, the decomposition is unique.

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21. A Matrix That Is Both Symmetric and Skew-Symmetric

Suppose A is both symmetric and skew-symmetric.

Then

$$A^T = A$$

and

$$A^T = -A.$$

Therefore,

$$A = -A.$$

Hence,

$$2A = 0.$$

Over the real numbers,

$$\boxed{A = 0.}$$

Thus, the **zero matrix is the only matrix that is both symmetric and skew-symmetric**.

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22. Useful Exam-Based Examples

Example 1

Determine whether

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$$

is symmetric.

We have

$$A^T = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} = A.$$

Therefore,

$$\boxed{A \text{ is symmetric.}}$$

Example 2

Determine whether

$$A = \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}$$

is skew-symmetric.

We have

$$A^T = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix} = -A.$$

Therefore,

$$\boxed{A \text{ is skew-symmetric.}}$$

Example 3

Find x and y if

$$A = \begin{pmatrix} 1 & x & 4 \\ 2 & 3 & y \\ 4 & 5 & 6 \end{pmatrix}$$

is symmetric.

For a symmetric matrix,

$$a_{12} = a_{21}.$$

Therefore,

$$x = 2.$$

Also,

$$a_{23} = a_{32}.$$

Therefore,

$$y = 5.$$

Hence,

$$\boxed{x = 2, \qquad y = 5.}$$

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23. Important Formulas

Transpose

$$\boxed{(A^T)^T = A}$$

$$\boxed{(A+B)^T = A^T + B^T}$$

$$\boxed{(AB)^T = B^T A^T}$$

$$\boxed{(kA)^T = kA^T}$$

Symmetric matrix

$$\boxed{A^T = A}$$

or

$$\boxed{a_{ij} = a_{ji}.}$$

Skew-symmetric matrix

$$\boxed{A^T = -A}$$

or

$$\boxed{a_{ij} = -a_{ji}.}$$

Decomposition

$$\boxed{A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)}$$

where

$$\boxed{\frac{1}{2}(A + A^T)}$$

is symmetric and

$$\boxed{\frac{1}{2}(A - A^T)}$$

is skew-symmetric.

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24. Practice Questions

1. Find the transpose of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

2. Verify that

$$(A + B)^T = A^T + B^T.$$

3. Verify that

$$(AB)^T = B^T A^T.$$

4. Determine whether

$$A = \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}$$

is symmetric.

5. Determine whether

$$A = \begin{pmatrix} 0 & 3 & -2 \\ -3 & 0 & 5 \\ 2 & -5 & 0 \end{pmatrix}$$

is skew-symmetric.

6. Find x, y, z such that

$$\begin{pmatrix} 1 & x & 2 \\ 3 & 4 & y \\ z & 6 & 7 \end{pmatrix}$$

is symmetric.

7. Show that the sum of two symmetric matrices is symmetric.
8. Show that the sum of two skew-symmetric matrices is skew-symmetric.
9. Express

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$$

as the sum of a symmetric matrix and a skew-symmetric matrix.

10. Prove that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Quick Revision

$$\boxed{\text{Transpose: } A^T = [a_{ji}]}$$

$$\boxed{\text{Symmetric: } A^T = A}$$

$$\boxed{\text{Skew-symmetric: } A^T = -A}$$

$$\boxed{\text{Diagonal entries of skew-symmetric matrix} = 0}$$

$$\boxed{A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)}$$