Lecture on Wronskian and Its Properties

Ordinary Differential Equations (ODE)

1 Introduction

The **Wronskian** is an important concept in the theory of ordinary differential equations. It is mainly used to determine whether a set of functions is linearly independent and to study the fundamental set of solutions of a linear differential equation.

Suppose

$$y_1(x), y_2(x), \dots, y_n(x)$$

are n differentiable functions. Their Wronskian is a determinant formed using the functions and their successive derivatives.

2 Definition of Wronskian

For two functions $y_1(x)$ and $y_2(x)$, the **Wronskian** is defined as

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Therefore,

$$\boxed{W(y_1, y_2) = y_1 y_2' - y_2 y_1'}$$

For three functions y_1, y_2, y_3,

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.$$

In general, for n functions $y_1, y_2, \dots, y_n$, the Wronskian is defined by

$$\boxed{W(y_1, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}}$$

3 Linear Dependence and Independence

Functions $y_1, y_2, \dots, y_n$ are said to be **linearly dependent** on an interval if there exist constants $c_1, c_2, \dots, c_n$, not all zero, such that

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0.$$

If the only solution is

$$c_1 = c_2 = \cdots = c_n = 0,$$

then the functions are said to be **linearly independent**.

Theorem 1. *If*

$$W(y_1, y_2, \ldots, y_n)(x_0) \neq 0$$

at some point x_0*, then*

$$y_1, y_2, \ldots, y_n$$

are linearly independent.

Proof

Suppose that

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0.$$

Differentiating this equation $0, 1, \ldots, n-1$ times, we obtain

$$\begin{aligned} c_1y_1 + c_2y_2 + \cdots + c_ny_n &= 0, \\ c_1y_1' + c_2y_2' + \cdots + c_ny_n' &= 0, \\ &\vdots \\ c_1y_1^{(n-1)} + c_2y_2^{(n-1)} + \cdots + c_ny_n^{(n-1)} &= 0. \end{aligned}$$

The coefficient determinant of this system is precisely the Wronskian. If

$$W(x_0) \neq 0,$$

the system has only the trivial solution

$$c_1 = c_2 = \cdots = c_n = 0.$$

Hence the functions are linearly independent.

$$\boxed{\text{Hence proved.}}$$

4 Wronskian of Two Functions

For two functions,

$$W(y_1, y_2) = y_1y_2' - y_2y_1'.$$

4.1 Example 1

Find the Wronskian of

$$y_1 = e^x, \qquad y_2 = xe^x.$$

We have

$$y_1' = e^x$$

and

$$y_2' = e^x + xe^x = (1+x)e^x.$$

Therefore,

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{vmatrix} \\ &= e^x(1+x)e^x - xe^xe^x \\ &= e^{2x}. \end{aligned}$$

Thus,

$$\boxed{W(y_1, y_2) = e^{2x} \neq 0.}$$

Therefore, e^x and xe^x are linearly independent.

5 Wronskian and Linear Dependence

Suppose

$$y_2 = cy_1,$$

where c is a constant. Then

$$y_2' = cy_1'.$$

Hence,

$$\begin{aligned} W(y_1, y_2) &= y_1y_2' - y_2y_1' \\ &= y_1(cy_1') - (cy_1)y_1' \\ &= 0. \end{aligned}$$

Therefore,

$$\boxed{\text{Linear dependence} \implies W = 0.}$$

However, for arbitrary differentiable functions, the converse is not always true. For solutions of a linear homogeneous differential equation under the usual continuity assumptions, Abel's theorem gives the stronger result discussed below.

6 Wronskian for a Second-Order Linear ODE

Consider the second-order homogeneous linear differential equation

$$\boxed{y'' + P(x)y' + Q(x)y = 0.}$$

Let y_1 and y_2 be two solutions.
Their Wronskian is

$$W = y_1y_2' - y_2y_1'.$$

Differentiating,

$$\begin{aligned} W' &= y_1'y_2' + y_1y_2'' - y_2'y_1' - y_2y_1'' \\ &= y_1y_2'' - y_2y_1''. \end{aligned}$$

Since y_1 and y_2 satisfy the differential equation,

$$y_i'' = -P(x)y_i' - Q(x)y_i, \qquad i = 1, 2.$$

Therefore,

$$\begin{aligned} W' &= y_1[-P(x)y_2' - Q(x)y_2] - y_2[-P(x)y_1' - Q(x)y_1] \\ &= -P(x)(y_1y_2' - y_2y_1'). \end{aligned}$$

Thus,

$$\boxed{W' + P(x)W = 0.}$$

7 Abel's Formula

From
$$W' + P(x)W = 0,$$
we have
$$\frac{W'}{W} = -P(x).$$

Integrating,
$$\ln|W| = -\int P(x)\,dx + C.$$

Hence,
$$\boxed{W(x) = Ce^{-\int P(x)\,dx}.}$$

More precisely, if x_0 is a fixed point, then
$$\boxed{W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} P(t)\,dt\right).}$$

This result is known as **Abel's formula** or **Abel–Liouville formula**.

8 Abel's Theorem

Theorem 2 (Abel's Theorem)**.** *Consider*
$$y'' + P(x)y' + Q(x)y = 0,$$
where P and Q are continuous on an interval I. If y_1 and y_2 are solutions, then
$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} P(t)\,dt\right).$$

Consequently,
$$W(x_0) = 0 \quad \Longrightarrow \quad W(x) = 0$$
throughout the interval.

Similarly,
$$W(x_0) \neq 0 \quad \Longrightarrow \quad W(x) \neq 0$$
throughout the interval.

9 Example Using Abel's Formula

Consider
$$y'' + \frac{2}{x}y' + y = 0.$$

Here,
$$P(x) = \frac{2}{x}.$$

Using Abel's formula,
$$W(x) = Ce^{-\int \frac{2}{x}\,dx}.$$

Therefore,

$$W(x) = Ce^{-2\ln|x|}$$
$$= \frac{C}{x^2}.$$

Hence,

$$\boxed{W(x) = \frac{C}{x^2}.}$$

10 Important Properties of the Wronskian

Property 1. If $y_1, \ldots, y_n$ are linearly dependent, then

$$\boxed{W(y_1, \ldots, y_n) = 0.}$$

Property 2. If

$$W(y_1, \ldots, y_n)(x_0) \neq 0,$$

then

$$\boxed{y_1, \ldots, y_n \text{ are linearly independent.}}$$

Property 3. Interchanging two functions changes the sign of the Wronskian:

$$\boxed{W(y_2, y_1) = -W(y_1, y_2).}$$

Property 4. If c is a constant, then

$$\boxed{W(cy_1, y_2) = cW(y_1, y_2).}$$

Similarly,

$$W(y_1, cy_2) = cW(y_1, y_2).$$

Property 5. If two columns of the Wronskian determinant are identical, then

$$\boxed{W = 0.}$$

Property 6. If y_1 and y_2 form a fundamental set of solutions of a second-order homogeneous linear ODE, then

$$\boxed{W(y_1, y_2) \neq 0.}$$

Property 7. For a second-order equation

$$y'' + P(x)y' + Q(x)y = 0,$$

the Wronskian satisfies

$$\boxed{W(x) = Ce^{-\int P(x)\,dx}.}$$

11 Examples of Common Wronskians

11.1 Example 2: 1 and x

Let

$$y_1 = 1, \qquad y_2 = x.$$

Then

$$y_1' = 0, \qquad y_2' = 1.$$

Thus,

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1.$$

Hence,

$$\boxed{W = 1 \neq 0.}$$

Therefore, 1 and x are linearly independent.

11.2 Example 3: e^x and e^{2x}

Let

$$y_1 = e^x, \qquad y_2 = e^{2x}.$$

Then

$$y_1' = e^x, \qquad y_2' = 2e^{2x}.$$

Therefore,

$$\begin{aligned} W &= e^x(2e^{2x}) - e^{2x}(e^x) \\ &= e^{3x}. \end{aligned}$$

Hence,

$$\boxed{W = e^{3x} \neq 0.}$$

Thus the functions are linearly independent.

11.3 Example 4: $\sin x$ and $\cos x$

Let

$$y_1 = \sin x, \qquad y_2 = \cos x.$$

Then

$$y_1' = \cos x, \qquad y_2' = -\sin x.$$

Therefore,

$$\begin{aligned} W &= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} \\ &= -\sin^2 x - \cos^2 x \\ &= -1. \end{aligned}$$

Thus,

$$\boxed{W = -1 \neq 0.}$$

Hence $\sin x$ and $\cos x$ are linearly independent.

12 Wronskian of Three Functions

For three functions,

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.$$

12.1 Example 5

Find the Wronskian of

$$y_1 = 1, \qquad y_2 = x, \qquad y_3 = x^2.$$

We have

$$y_1' = 0, \qquad y_1'' = 0,$$

$$y_2' = 1, \qquad y_2'' = 0,$$

and

$$y_3' = 2x, \qquad y_3'' = 2.$$

Therefore,

$$W = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}.$$

Since the determinant is upper triangular,

$$W = 1(1)(2) = 2.$$

Hence,

$$\boxed{W = 2 \neq 0.}$$

Therefore,

$$1, \quad x, \quad x^2$$

are linearly independent.

13 Wronskian and Fundamental Set of Solutions

Suppose

$$y'' + P(x)y' + Q(x)y = 0.$$

If y_1 and y_2 are linearly independent solutions, then every solution can be written as

$$\boxed{y = c_1 y_1 + c_2 y_2.}$$

The pair

$$\{y_1, y_2\}$$

is called a **fundamental set of solutions**.

For such a fundamental set,

$$\boxed{W(y_1, y_2) \neq 0.}$$

14 Wronskian and Initial Conditions

Suppose

$$y'' + P(x)y' + Q(x)y = 0$$

has two solutions y_1, y_2.

If

$$W(x_0) \neq 0,$$

then the matrix

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix}$$

is invertible.

Therefore, for arbitrary initial conditions

$$y(x_0) = a, \qquad y'(x_0) = b,$$

there exist unique constants c_1, c_2 such that

$$y = c_1 y_1 + c_2 y_2.$$

Thus, the Wronskian is closely related to the uniqueness of solutions of linear initial-value problems.

15 Wronskian and Fundamental Matrix

For a second-order equation, define the fundamental matrix

$$Y(x) = \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}.$$

Then

$$\det Y(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Therefore,

$$\boxed{\det Y(x) = W(y_1, y_2)(x).}$$

Hence, if

$$W(x) \neq 0,$$

the fundamental matrix is nonsingular.

16 Wronskian for an n-th Order ODE

Consider the normalized n-th order homogeneous linear differential equation

$$\boxed{y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0.}$$

If

$$y_1, y_2, \ldots, y_n$$

are solutions, then Abel's identity gives

$$\boxed{W(x) = Ce^{-\int a_{n-1}(x)\,dx}.}$$

This generalizes Abel's formula for second-order equations.

17 Wronskian in Variation of Parameters

Consider the nonhomogeneous equation

$$y'' + P(x)y' + Q(x)y = R(x).$$

Let y_1, y_2 be linearly independent solutions of the corresponding homogeneous equation.

The Wronskian is

$$W = y_1 y_2' - y_2 y_1'.$$

In the method of variation of parameters,

$$\boxed{u_1' = -\frac{y_2 R}{W}}$$

and

$$\boxed{u_2' = \frac{y_1 R}{W}.}$$

The particular solution is then

$$\boxed{y_p = u_1 y_1 + u_2 y_2.}$$

Thus, the Wronskian plays an essential role in the method of variation of parameters.

18 Practice Questions

1. Find the Wronskian of 1 and x^2.
2. Find the Wronskian of e^x and e^{-x}.
3. Find the Wronskian of $\sin x$ and $\cos x$.
4. Determine whether x and x^2 are linearly independent using the Wronskian.
5. Find the Wronskian of $1, x, x^2$.
6. If y_1, y_2 are solutions of
$$y'' + 3y' + 2y = 0,$$
find the form of their Wronskian.
7. For
$$y'' + \frac{1}{x}y' + y = 0,$$
determine $W(x)$ using Abel's formula.
8. Show that e^x and xe^x are linearly independent.
9. Show that e^x and $2e^x$ are linearly dependent using the Wronskian.
10. Explain the role of the Wronskian in the method of variation of parameters.

19 Quick Revision

Concept	Key Result
Two-function Wronskian	$W = y_1 y_2' - y_2 y_1'$
Non-zero Wronskian	Implies linear independence
Second-order ODE	$W' + P(x)W = 0$
Abel's formula	$W = Ce^{-\int P(x)\,dx}$
Fundamental solutions	$W \neq 0$ throughout the interval
Variation of parameters	Uses W in the formulas for u_1' and u_2'

20 Conclusion

The Wronskian is a powerful tool in the study of ordinary differential equations. Its principal applications are:

- testing linear independence of solutions,
- identifying a fundamental set of solutions,
- deriving Abel's formula,
- studying uniqueness of initial-value problems, and
- solving nonhomogeneous equations by variation of parameters.

For a second-order homogeneous linear ODE,

$$\boxed{y'' + P(x)y' + Q(x)y = 0,}$$

the most important formula to remember is

$$\boxed{W(x) = Ce^{-\int P(x)\,dx}.}$$