

Triple Integrals in Cylindrical Coordinates

In this section we want to take a look at triple integrals done completely in Cylindrical Coordinates. Recall that cylindrical coordinates are really nothing more than an extension of polar coordinates into three dimensions. The following are the conversion formulas for cylindrical coordinates:

$$\begin{aligned}x &= r \cos(\theta) \\y &= r \sin(\theta) \\z &= z\end{aligned}$$

In order to do the integral in cylindrical coordinates we will need to know what dV will become in terms of cylindrical coordinates. We will be able to show in the Change of Variables section of this chapter that:

Fact

$$dV = r \, dz \, dr \, d\theta$$

The region, E , over which we are integrating becomes,

$$\begin{aligned}E &= \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\} \\&= \{(r, \theta, z) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta), u_1(r \cos \theta, r \sin \theta) \leq z \leq u_2(r \cos \theta, r \sin \theta)\}\end{aligned}$$

Note that we've only given this for E in which D is in the xy -plane. We can modify this accordingly if D is in the yz -plane or the xz -plane as needed.

In terms of cylindrical coordinates a triple integral is:

Fact

$$\iiint_E f(x, y, z) \, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} r f(r \cos \theta, r \sin \theta, z) \, dz \, dr \, d\theta$$

Don't forget to add in the r and make sure that all the x 's and y 's also get converted over into cylindrical coordinates.

Let's see an example.

Example 1

Evaluate $\iiint_E y \, dV$ where E is the region that lies below the plane $z = x + 2$, above the xy -plane, and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution There really isn't too much to do with this one other than do the conversions and then evaluate the integral.

We'll start out by getting the range for z in terms of cylindrical coordinates:

$$0 \leq z \leq x + 2 \implies 0 \leq z \leq r \cos(\theta) + 2$$

Remember that we are above the xy -plane and so we are above the plane $z = 0$.

Next, the region D is the region between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the xy -plane and so the ranges for it are:

$$0 \leq \theta \leq 2\pi, \quad 1 \leq r \leq 2$$

Here is the integral:

$$\begin{aligned} \iiint_E y \, dV &= \int_0^{2\pi} \int_1^2 \int_0^{r \cos(\theta) + 2} (r \sin(\theta)) r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_1^2 r^2 \sin(\theta) (r \cos(\theta) + 2) \, dr \, d\theta \\ &= \int_0^{2\pi} \int_1^2 \left(\frac{1}{2} r^3 \sin(2\theta) + 2r^2 \sin(\theta) \right) \, dr \, d\theta \\ &= \int_0^{2\pi} \left(\frac{1}{8} r^4 \sin(2\theta) + \frac{2}{3} r^3 \sin(\theta) \right) \Big|_1^2 \, d\theta \\ &= \int_0^{2\pi} \left(\frac{15}{8} \sin(2\theta) + \frac{14}{3} \sin(\theta) \right) \, d\theta \\ &= \left(-\frac{15}{16} \cos(2\theta) - \frac{14}{3} \cos(\theta) \right) \Big|_0^{2\pi} \\ &= 0 \end{aligned}$$

Just as we did with double integrals involving polar coordinates, we can start with an iterated integral in terms of x , y , and z and convert it to cylindrical coordinates.

Example 2

Convert $\int_{-1}^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy$ into an integral in cylindrical coordinates.

Solution Here are the ranges of the variables from this iterated integral:

$$\begin{aligned} -1 &\leq y \leq 1 \\ 0 &\leq x \leq \sqrt{1-y^2} \\ x^2 + y^2 &\leq z \leq \sqrt{x^2 + y^2} \end{aligned}$$

The first two inequalities define the region D . Since the upper and lower bounds for x are $x = \sqrt{1-y^2}$ and $x = 0$, we know that we've got at least part of the right half of a circle of radius 1 centered at the origin. Since the range of y 's is $-1 \leq y \leq 1$, we know that we have the complete right half of the disk of radius 1 centered at the origin. So, the ranges for D in cylindrical coordinates are:

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 1$$

All that's left to do now is to convert the limits of the z -range:

$$r^2 \leq z \leq r$$

On a side note, notice that the lower bound here is an elliptic paraboloid and the upper bound is a cone. Therefore, E is a portion of the region between these two surfaces.

The integral is:

$$\begin{aligned} \int_{-1}^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy &= \int_{-\pi/2}^{\pi/2} \int_0^1 \int_{r^2}^r r(r \cos \theta)(r \sin \theta) z \, dz \, dr \, d\theta \\ &= \int_{-\pi/2}^{\pi/2} \int_0^1 \int_{r^2}^r zr^3 \cos(\theta) \sin(\theta) \, dz \, dr \, d\theta \end{aligned}$$