●The transformation relationship between Cartesian and Spherical coordinates system can be expressed as

Cartesian to Spherical	Spherical to Cartesian
$x = r\sin\theta\cos\varphi$	$r = \sqrt{x^2 + y^2 + z^2}$
$y = r\sin\theta\sin\varphi$	$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$
$z = r\cos\theta$	$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$

●Generalized curvilinear coordinate system is simply a general way to represents all coordinate systems (Cartesian, Cylindrical and spherical etc) which may be orthogonal and nonorthogonal.

●The scale factors, variables and unit vectors for three coordinate systems (Cartesian, Cylindrical and Spherical) can be tabulated as table.1.

●Differential volume element for different coordinate systems can be expressed as

Coordinate System	**Volume Element**
Curvilinear	$h_1 h_2 h_3 \; du_1 du_2 du_3$
Cartesian	$dx dy dz$
Cylindrical	$r\,dr\,d\varphi\,dz$
Spherical	$r^2 \sin\theta\,dr\,d\theta\,d\varphi$

● Gradient for different coordinate systems can be expressed as

Coordinate System	**Gradient**
Orthogonal Curvilinear	$\text{grad } f = \nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{a}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{a}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{a}_3$
Cartesian	$\text{grad } f = \frac{\partial f}{\partial x}\hat{a}_x + \frac{\partial f}{\partial y}\hat{a}_y + \frac{\partial f}{\partial z}\hat{a}_z$
Cylindrical	$\text{grad } f = \frac{\partial f}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\hat{a}_\varphi + \frac{\partial f}{\partial z}\hat{a}_z$
Spherical	$\text{grad } f = \frac{\partial f}{\partial u_1}\hat{a}_r + \frac{1}{r}\frac{\partial f}{\partial u_2}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial d\varphi}\hat{a}_\varphi$

- Divergence for different coordinate systems can be expressed as

Coordinate System	**Divergence**
Orthogonal Curvilinear	$\operatorname{div}A = \dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial}{\partial u_1}\left(A_1h_2h_3\right)+\dfrac{\partial}{\partial u_2}\left(A_2h_1h_3\right)+\dfrac{\partial}{\partial u_3}\left(A_3h_1h_2\right)\right]$
Cartesian	$\operatorname{div}A = \left[\dfrac{\partial}{\partial x}\left(A_x\right)+\dfrac{\partial}{\partial y}\left(A_y\right)+\dfrac{\partial}{\partial z}\left(A_z\right)\right]$
Cylindrical	$\operatorname{div}A = \dfrac{1}{r}\left[\dfrac{\partial}{\partial r}\left(A_r r\right)+\dfrac{\partial}{\partial \varphi}\left(A_\varphi\right)+\dfrac{\partial}{\partial z}\left(A_z r\right)\right]$
Spherical	$\operatorname{div}A = \dfrac{1}{r^2\sin\theta}\left[\dfrac{\partial}{\partial r}\left(A_r r^2\sin\theta\right)+\dfrac{\partial}{\partial \theta}\left(A_\theta r\sin\theta\right)+\dfrac{\partial}{\partial \varphi}\left(A_\varphi r\right)\right]$

- In orthogonal curvilinear co-ordinates system curlA can be written as

$$\operatorname{curl}A = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{a}_1 & h_2\hat{a}_2 & h_3\hat{a}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix}$$

- In orthogonal curvilinear co-ordinates system Laplacian vector $\nabla^2 f$ can be written as

$$\nabla^2 f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial u_1}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$$

2.13 Glossary

Orthogonal coordinate system: When the surfaces intersect perpendicularly we have an orthogonal coordinate system.

Curvilinear co-ordinates: The value of u_1,u_2,u_3 for the three surfaces intersecting at P are called ***curvilinear co-ordinates*** or curvilinear surfaces

2.14 Answers to Self Learning Exercises

Answers to Self Learning Exercise -I

Ans.1 : (0,0,0) **Ans.2 :** $(0 \le \varphi \le 2\pi)$

Ans.3 : René Descartes **Ans.4 :** $dv = dx\,dy\,dz$

Ans.5 : $dv = \rho\,d\rho\,d\varphi\,dz$ **Ans.6 :** $dv = r^2 \sin\theta\,dr\,d\theta\,d\varphi$

Answers to Self Learning Exercise -II

Ans.1 : Conical **Ans.2 :** Plane

Ans.3 : A circular cylinder **Ans.4 :** $\sqrt{5}$ units

Ans.5 : $\sqrt{13}$ units **Ans.6 :** $2\sqrt{5+2\sqrt{2}}$ units

2.15 Exercises

Section-A (Very Short Answer Type Questions)

Q.1 In orthogonal curvilinear coordinate system three axes are ………. to each other.

Q.2 Write the formula to determine the base vector for coordinates system?

Q.3 In which coordinate system uses two angles and one distance?

Q.4 In which coordinate system uses two distances and one angle?

Q.5 In which coordinate system uses only distance?

Section-B (Short Answer Type Questions)

Q.6 Compute the vector directed from (1,1,1) to (2,2,2) in Cartesian coordinates system.

Q.7 Determine the value of $\nabla\vec{A}$ at point (1,-1,1). If $\vec{A} = xy\,\hat{i} - x^2z\,\hat{j} + z\hat{k}$

Q.8 Find the location of the point (1,2,3) in cylindrical coordinates system.

Q.9 Write the formula of the base vectors for a coordinate system.

Q.10 Find the location of the point (1,2,3) in spherical coordinates system.

Section C (Long Answer Type Questions)

Q.11 Determine the base vectors for the cylindrical coordinate system.

Q.12 Determine the base vectors for the spherical coordinate system.

Q.13 Derive the expressions for the distance between two points in the cylindrical and spherical coordinate systems.

Q.14 Evaluate the transformation relationship between cylindrical to spherical coordinate system.

Q.15 Transform the vector $\vec{A} = xy\,\hat{i} - x^2 z\,\hat{j} + z\hat{k}$ from Cartesian coordinates to cylindrical and spherical coordinate systems.

2.16 Answers to Exercise

Ans.1 : Mutually Perpendicular

Ans.2 : Plane

Ans.3: Spherical

Ans.4 : Cylindrical

Ans.5 : Cartesian

Ans.6 : $\vec{A} = \hat{a}_x + \hat{a}_y + \hat{a}_z$

Ans.7 : -1

Ans.8 : $r = \sqrt{5}$ units, $\varphi = 63^\circ 43'$, $z = 3$

Ans.9 : $\vec{b}_i = \dfrac{\partial \vec{R}}{\partial u_i} \qquad i = 1,2,3$

Ans.10 : $r = \sqrt{14}$ units, $\theta = 0.99^\circ$, $\varphi = 63^\circ 43'$

References and Suggested Readings

1. K.D. Prasad, 'Electromagnetic Fields and Waves', 1st edition, Satya Prakashan, New Delhi (1999).
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3. B.S. Rajput, Mathematical Physics, 1st edition,Pragati Prakashan, Meerut (INDIA).
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UNIT-3

Gauss's theorem, Stokes's theorem

Structure of the Unit

3.0 Objectives

After gone through this unit learner will able to solve any physical problem in which vector integration is used. Learner can apply Gauss divergence theorem & convert surface integral into volume integral.

3.1 Introduction

In this unit integral calculus part of vector calculus is discussed. Vector line integral, vector surface integral & volume integral are explained. Using of Gauss divergence theorem & Stoke theorem with various examples are explained.

3.2 Line Integrals

Suppose a continuous vector function $F(x, y, z)$ defined at each point of the curve C

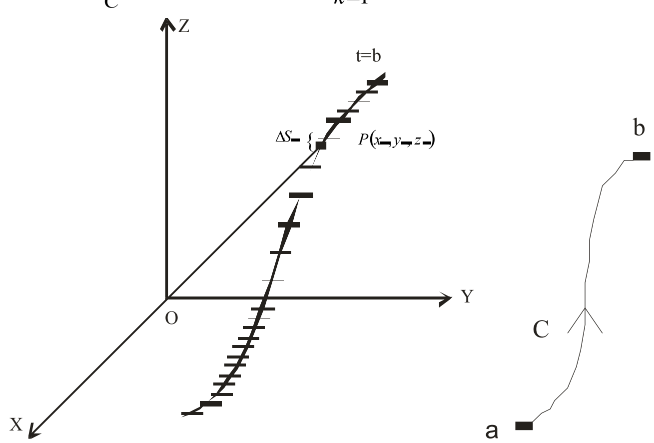
$$r(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}, \; a \le t \le b.$$

We partition the curve into a finite number of sub arcs. The typical sub arc has length Δs_k in each sub arc we choose we point (x_k, y_k, z_k) and from the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\, \Delta s_k \qquad (1)$$

The sum in (1) approaches a limit as n increases, and the length Δs_k approach zero. We call this limit the **integral of F over the Curve *C* from *a* to *b*.**

$$\therefore \qquad \int_C f(x, y, z) = \lim_{n \to \infty} \sum_{k=1}^{n} F(x_k, y_k, z_k)\, \Delta s_k \qquad (2)$$



Let $F(r)$ be a continuous vector function, then component of $F(r)$ along the tangent at P is

$$F(r).\frac{dr}{ds} \qquad \left(\because \frac{dr}{ds} \text{ is unit tangent vector at P}\right)$$

and $\int_C \left(F(r) . \frac{dr}{ds} \right)$ or $\int_C f(r) . dr$

is called the tangent line integral of $F(r)$ along the curve C.

Let $F(r) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$r = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore \quad \int_C \vec{F}.d\vec{r} = \int_C \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k} \right) . \left(dx\hat{i} + dy\hat{j} + dz\hat{k} \right)$$

$$= \int_C \left(F_1 dx + F_2 dy + F_3 dz \right) = \int_C \left[F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right] dt$$

$$= \int_C F(r) . \frac{dr}{dt} dt$$

3.3 Properties of the Line Integral

Let F and G be two continuous vector point function and k is any constant, then

1. $\int_C k\vec{F}.d\vec{r} = k\int_C \vec{F}.d\vec{r}$

2. $\int_C \left(\vec{F} + \vec{G} \right) . d\vec{r} = \int_C \vec{F}.d\vec{r} + \int_C \vec{G}.d\vec{r}$

3. $\int_C \vec{F}.d\vec{r} = -\int_{C_1} \vec{F}.d\vec{r}$

   Where direction of C_1 is opposite to curve C.

4. $\int_C \vec{F}.d\vec{r} = \int_{C_1} \vec{F}.d\vec{r} + \int_{C_2} \vec{F}.d\vec{r}$

5. ***If the line integral depends only on the end points of the curve, not on the path joining them then vector field is called conservative vector field***.

Let $F = \nabla\phi$; F is conservative field and ϕ is its scalar potential.

$$\int_C \vec{F}.d\vec{r} = \int_a^b \vec{F}.d\vec{r} = (\phi)_a^b$$

$$= \int_a^b \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} \right) . \left(dx\hat{i} + dy\hat{j} + dz\hat{k} \right)$$

$$= \int_a^b \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right)$$

$$= \int_a^b d\phi = [\phi]_a^b \;\; = \phi(b) - \phi(a)$$

Thus, if the curve is closed, then the ***line integral of conservative vector field*** $F(r)$ ***is zero*** i.e. $\oint_C F.dr = 0$.

3.4 Application of the Line Integral

(a) Circulation: If F represent the velocity of a fluid and C is a closed curve, then the integral $\oint_C F.dr$ is called circulation of F around the curve C i.e.

$$\text{Circulation } = \oint_C F.dr$$

(b) Work done by a Force : If F represents the force acting on a particle moving along an arc AB, then the work done by the force F during the displacement from A to B is

$$\text{Work done } = \int_A^B F.dr$$

If F is a conservative vector field and ϕ is scalar potential of F, then

$$\text{Work done } = \int_A^B F.dr$$

$$= \int_A^B \nabla\phi.dr$$

$$= \phi(B) - \phi(A)$$

If curve is closed, then work done $\oint F.dr = 0$.

Example 1 Find the total work done in moving a particle in a force field given by $F = 3xy\hat{i} - 5y\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$

Sol. Total work done $= \int_C F.dr = \int_C \left(3xy\hat{i} - 5z\hat{j} + 10x\hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$

Since $x = t^2 + 1$, $y = 2t^2$ and $z = t^3$

$$\therefore \qquad dr = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$= 2tdt\hat{i} + 4tdt\hat{j} + 3t^2dt\hat{k}$$

$$\text{Now work done } = \int_1^2 \left[3\left(t^2+1\right)2t^2\hat{i} - 5t^3\hat{j} + 10\left(t^2+1\right)\hat{k}\right].\left[2tdt\hat{i} + 4tdt\hat{j} + 3t^2dt\hat{k}\right]$$

$$= \int_1^2 \left[12t^3\left(t^2+1\right) - 20t^4 + 30t^2\left(t^2+1\right)\right] dt$$

$$= \int_1^2 \left(12t^3 + 10t^4 + 12t^3 + 30t^2\right) dt$$

$$= \int_1^2 \left[12.\frac{t^6}{6} + 10.\frac{t^5}{5} + 12.\frac{t^4}{4} + 30.\frac{t^3}{3}\right]_1^2 = 303 \text{ units.}$$

Example 2 If $F = \left(2x + y^2\right)\hat{i} + \left(2y - 4x\right)\hat{j}$. Evaluate $\oint_C F.dr$ around a triangle ABC in the xy-plane with A(0, 0), B(2, 0), C(2, 1) in counter clockwise direction. What is its value in clockwise direction.

Sol. The curve C is union of three curves C_1, C_2 and C_3

$$\oint_C F.dr = \int_{C_1} F.dr + \int_{C_2} F.dr + \int_{C_3} F.dr$$

$$= I_1 + I_2 + I_3 \text{ (say)}$$

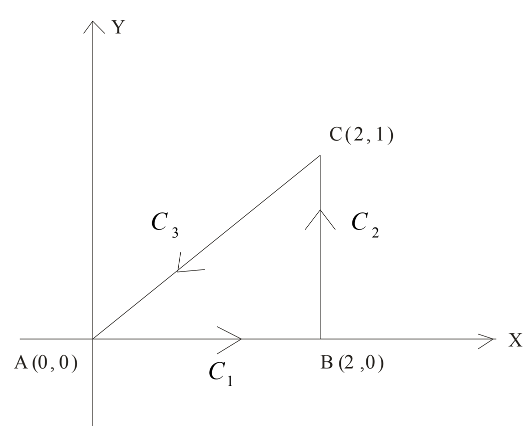
Along C_1: Straight line AB, y=0, z=0 and x varies from 0 to 2.

$$\therefore r = xi \Rightarrow dr = dxi$$

$$I_1 = \int_C F.dr = \int_{C_1} \left[\left(2x + y\right)i + \left(3y - 4x\right)j\right].\left(dxi\right)$$

$$= \int_{C_1} \left(2x + y^2\right)dx = \int_0^2 2xdx$$

$$= \left(x^2\right)_0^2 = 4$$



Along C_2 : The straight line BC, x=2, z=0 and y varies from 0 to 1.

$$\therefore r = 2i + yj \Rightarrow dr = dy\, j$$

Thus, $I_2 = \int_{C_2} F.dr = \int_{C_2} (3y - 4x)\, j.dy\, j$

$$= \int_0^1 (3y - 8)dy$$

$$= \left(\frac{3}{2}y^2 - 8y\right)_0^1 = \frac{3}{2} - 8 = \frac{-13}{2}$$

Along C_3 : The straight line CA, z = 0, 2y = x and x varies from 2 to 0

$$\therefore r = xi + \frac{x}{2}j + 0k \Rightarrow dr = dxi + \frac{dx}{2}j = \left(i + \frac{j}{2}\right)dx$$

Thus, $I_3 = \int_C F.dr = \int_C \left[\left(2x + y^2\right)i + (3y - 4x)j\right].\left(i + \frac{j}{2}\right)dx$

$$= \int_{C_3} \left\{\left(2x + y^2\right) + \frac{1}{2}(3y - 4x)\right\}dx$$

$$= \int_2^0 \left(2x + \frac{x^2}{4} + \frac{3}{4}x - \frac{4x}{2}\right)dx$$

$$= \left(\frac{x^3}{12} + \frac{3x^2}{8}\right)_2^0 = -\frac{8}{12} - \frac{12}{8} = \frac{-13}{6}$$

The required integral in counter clockwise direction is

$$\oint_C F.dr = I_1 + I_2 + I_3 = 4 - \frac{13}{2} - \frac{13}{6} = \frac{-14}{3}$$

The value of the integral in clockwise direction.

$$\int_{C_1} F.dr = -\int_C F.dr = \frac{14}{3}$$

3.5 Surface Integral

Let R is the shadow region of a surface S defined by the equation $f(x, y, z) = C$ and g is a continuous function defined at the points of S, then the integral of g over S is the integral

$$\iint_S g(x,y,z)dS = \iint_R g(x,y,z)\frac{|\nabla f|}{|\nabla f .|}dA \qquad \ldots (3)$$

Where P is a unit vector normal to R and $\nabla f.P \neq 0$. The integral itself is called a surface integral.

3.6 Surface Integral for Flux

Suppose that F is a continuous vector field defined over a two-sided surface S and $\hat{n}$ is the unit normal field on the surface. The integral of $F.\hat{n}$ over S is called the flux across S in the positive direction

$$\therefore \qquad \text{Flux } = \int\int_S F.\hat{n}dS$$

$$= \iint_R \left(F.\frac{\nabla g}{|\nabla g|}\right)\frac{|\nabla g|}{|\nabla g.P|}dA$$

$$= \iint_R F.\frac{\nabla g}{|\nabla g.P|}dA \qquad \ldots (4)$$

Example 3 : Find the flux of the vector field

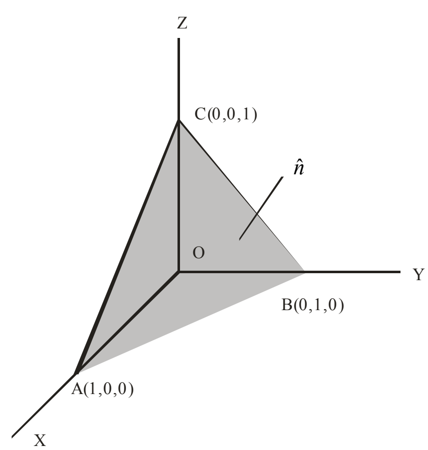
$A = (x-2z)\hat{i} + (x+3y+z)\hat{j} + (5x+y)\hat{k}$ through the upper side of the triangle ABC with vertices at the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1).

Sol. Equation of the plane containing the give triangle ABC is

$$f(x,y,z) \equiv x+y+z=1$$

Unit normal $\hat{n}$ to ABC is

$$\boxed{\hat{n} = \frac{\nabla f}{|\nabla f|}} = \frac{i+j+k}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}(i+j+k)$$



Flux of $A = \iint_S A.\hat{n}ds = \iint_S A.\hat{n}\frac{dA}{|\hat{n}.k|}$

$$= \iint_{AOB} [(x-2)i + (x+3y+z)j + (5x+y)k].\frac{(i+j+k)}{\sqrt{3}}.\frac{dxdy}{\frac{1}{\sqrt{3}}}$$

$$= \int_0^1 \int_0^{1-x} (x-2z+x+3y+z+5x+y)dxdy$$

$$= \int_0^1 \int_0^{1-x} [7x+4y-(1-x-y)]dxdy$$

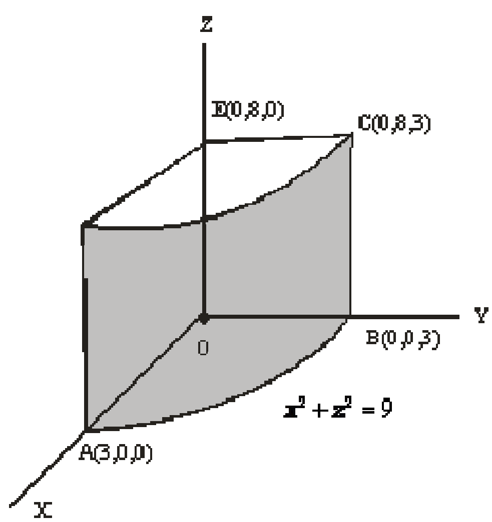
$$= \int_0^1 \left[(8x-1)y + \frac{5y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1 \left[(8x-1)(1-x) + \frac{5}{2}(1-x^2)\right] dx$$

$$= \left[\frac{-11}{2}\frac{x^3}{3} + 2x^2 + \frac{3}{2}x\right]_0^1 = \frac{-11}{6} + 2 + \frac{3}{2} = \frac{5}{3}$$

Example 4 : Evaluate $\iint_S A.\hat{n}ds$ over the entire surface S of the region bounded by the cylinder $x^2 + z^2 = 9$, x=0, y=0, z=0 and y=8 where $A = 6zi + (2x+y)i - xk$.

Sol. Here the entire surface S consist of 5 surfaces, namely. S_1 : lateral surface of the cylinder ABCD, S_2 : AOED, S_3 : OBCE, S_4 : OAB, S_5 : CDE.



Thus, $\iint_S A \,.\, \hat{n} dS = \int\int_{S_1+S_2+S_3+S_4+S_5} A \,.\, \hat{n} dS$

$$= \int\int_{S_1} A \,.\, \hat{n} dS + \int\int_{S_2} A \,.\, \hat{n} dS + ... + \int\int_{S_5} A \,.\, \hat{n} dS +$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 \text{ (say)}$$

S_1 : **ABCD :** The curved surface S_1 is $f = x^2 + z^2 = 9$. The unit outward normal to S_1 is

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2xi + 2zk}{\sqrt{4x^2 + 4z^2}} = \frac{xi + zk}{3}$$

$$\therefore \quad A.\hat{n} = [6zi + (2x+y)j - zk]k.\frac{(xi+zj)}{3}$$

$$= \frac{1}{3}(6xz - xz) = \frac{5}{3}xz$$

$$\text{and} \quad n.\hat{k} = \frac{z}{3}$$

$$\therefore \iint_{S_1} A.\hat{n}dS = \int_{y=0}^{8}\int_{x=0}^{3} \frac{5}{3}xz\frac{dxdy}{z/3}$$

$$= 5\int_0^8\int_0^3 xdxdy = \frac{5\times 9\times 8}{2} = 180$$

S_2 : **AOED :** The surface S_2 is xy-plane i.e. z=0. Unit outward normal to the surface is $\hat{n} = -k$

$$\therefore \iint_{S_2} A.\hat{n}dS = \iint [6zi + (2x+y)j - xk].(-k)\frac{dxdy}{|-k.k|}$$

$$= \int_{y=0}^{8}\int_{x=o}^{3} x\,dxdy$$

$$= \left(\frac{x^2}{2}\right)_0^3.(y)_0^8 = \frac{9}{2}\times 8 = 36$$

S_3 : **OBCE :** Surface S_3 is yz-plane i.e. x=0. Unit outward normal to S_3 is $\hat{n} = -i$.

$$\therefore \iint_{S_3} A.\hat{n}dS = \int_{z=0}^{3}\int_{y=0}^{8} [6zi + (2x+y)j - xk].(-i)\frac{dydz}{|-i.i|}$$

$$= \int_0^3 \int_0^8 -6z\,dydz = -6\left(\frac{z^2}{2}\right)_0^3 (y)_0^8 = -216$$

S_4 : **OAB :** The section OAB is in xz-plane i.e. y=0. The unit outward normal to S_4 is $\hat{n} = -j$.

$$\therefore \iint_{S_4} A.\hat{n}dS = \int_0^3 \int_0^{\sqrt{9-x^2}} [6zi + (2x+y)j - xk].(-j)\frac{dxdz}{|-j.j|}$$

$$= \int_0^3 \int_0^{\sqrt{9-x^2}} -2x\,dxdz$$

$$= \int_0^3 -2x(z)_0^{\sqrt{9-x^2}}\,dx$$

$$= \int_0^3 -2x\sqrt{9-x^2}\,dx$$

$$= \left[\frac{(9-x^2)^{3/2}}{\frac{3}{2}}\right] = \frac{2}{3}(-27) = -18$$

S_5 : **CDE :** The section S_5 is parallel to xz-plane, y=8. The outward normal to S_5 is $\hat{n} = j$.

$$\iint_{S_5} A.\hat{n}dS = \int_0^3 \int_0^{\sqrt{9-x^2}} [6zi + (2x+y)j - xk].(j)\frac{dxdz}{|j.j|}$$

$$= \int_0^3 \int_0^{\sqrt{9-x^2}} (2x+8)dxdz$$

$$= \int_0^3 (2x+8)(z)_0^{\sqrt{9-x^2}}\,dx$$

$$= 2\int_0^3 (x+4)\sqrt{9-x^2}\,dx$$

$$= 2\left[\frac{\left(9-x^2\right)^{3/2}}{3/2.(-2)} + 4.\frac{x}{2}\sqrt{9-x^2} + 4.\frac{9}{2}sin^{-1}\frac{x}{3}\right]_0^1$$

$$= 18(1+\pi)$$

Thus, the required surface integral is

$$\iint_S A.\hat{n}dS = 180 + 36 - 216 - 18 + 18 + 18\pi = 18\pi$$

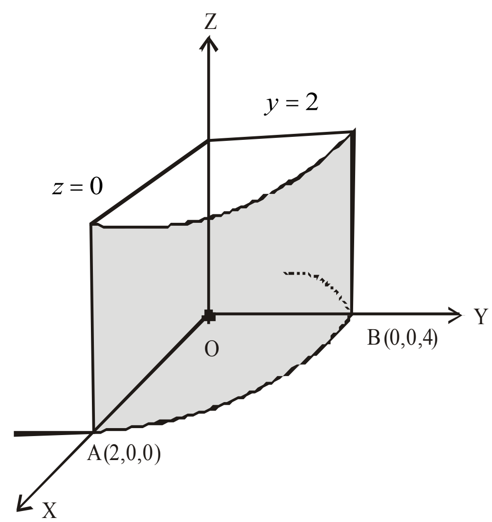
3.7 Volume Integral

Let V be a region in space enclosed by a closed surface S. Let ϕ be a scalar point function and F be a vector point function, then the triple integral

$$\iiint_V \phi\, dV \text{ and } \iiint_V F dV$$

and called volume integrals.

Example 5 : Evaluate $\iiint_V f dV$ where $f = 2x + y$, V is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 2$ and $z = 0$.



Sol. $$\iiint_V f dV = \int_{y=0}^{2}\int_{x=0}^{2}\int_{z=0}^{4-x^2}(2x - y)dzdxdy$$

$$= \int_{y=0}^{2}\int_{x=0}^{2}(2xz + yz)_0^{4-x^2} dxdy$$