1 Introduction

Hermitian and skew-Hermitian matrices are important classes of complex square matrices. They are the complex-number analogues of symmetric and skew-symmetric matrices, respectively.
Let

$$A = [a_{ij}]$$

be a complex square matrix. The **conjugate transpose** (or adjoint) of A is denoted by A^* and is defined by

$$\boxed{A^* = \overline{A}^T}.$$

Thus,

$$\boxed{(A^*)_{ij} = \overline{a_{ji}}}.$$

Here, $\overline{z}$ denotes the complex conjugate of the complex number z.

2 Hermitian Matrix

Definition 2.1. *A square complex matrix A is called a **Hermitian matrix** if*

$$\boxed{A^* = A}.$$

Equivalently,

$$\boxed{a_{ij} = \overline{a_{ji}}}.$$

Thus, the entries on opposite sides of the main diagonal are complex conjugates of each other.

2.1 Example

Consider

$$A = \begin{pmatrix} 2 & 1+i & 3-2i \\ 1-i & 4 & 5+i \\ 3+2i & 5-i & 6 \end{pmatrix}.$$

Taking the conjugate transpose,

$$A^* = \begin{pmatrix} 2 & 1+i & 3-2i \\ 1-i & 4 & 5+i \\ 3+2i & 5-i & 6 \end{pmatrix}.$$

Therefore,

$$A^* = A.$$

Hence, A is a Hermitian matrix.

3 General Form of a Hermitian Matrix

A 2×2 Hermitian matrix has the form

$$\boxed{A = \begin{pmatrix} a & z \\ \overline{z} & d \end{pmatrix}}$$

where $a, d \in \mathbb{R}$ and $z \in \mathbb{C}$.

For example,

$$A = \begin{pmatrix} 3 & 2+4i \\ 2-4i & 5 \end{pmatrix}$$

is Hermitian.

3.1 Important Property

The diagonal elements of a Hermitian matrix are always real.

Indeed,

$$a_{ii} = \overline{a_{ii}},$$

which implies

$$\boxed{a_{ii} \in \mathbb{R}}.$$

4 Skew-Hermitian Matrix

Definition 4.1. *A square complex matrix A is called a **skew-Hermitian matrix** if*

$$\boxed{A^* = -A}.$$

Equivalently,

$$\boxed{a_{ij} = -\overline{a_{ji}}}.$$

4.1 Example

Consider

$$A = \begin{pmatrix} 2i & 1+i \\ -1+i & -3i \end{pmatrix}.$$

Its conjugate transpose is

$$A^* = \begin{pmatrix} -2i & -1-i \\ 1-i & 3i \end{pmatrix}.$$

But

$$-A = \begin{pmatrix} -2i & -1-i \\ 1-i & 3i \end{pmatrix}.$$

Therefore,

$$A^* = -A.$$

Hence, A is a skew-Hermitian matrix.

5 General Form of a Skew-Hermitian Matrix

A 2×2 skew-Hermitian matrix has the form

$$\boxed{A = \begin{pmatrix} ia & z \\ -\overline{z} & id \end{pmatrix}}$$

where $a, d \in \mathbb{R}$ and $z \in \mathbb{C}$.

For example,

$$A = \begin{pmatrix} 2i & 3+i \\ -3+i & -5i \end{pmatrix}$$

is skew-Hermitian.

5.1 Important Property

The diagonal elements of a skew-Hermitian matrix are purely imaginary, including zero.

Since

$$a_{ii} = -\overline{a_{ii}},$$

we obtain

$$\boxed{\mathrm{Re}(a_{ii}) = 0}.$$

6 Comparison with Symmetric and Skew-Symmetric Matrices

For real matrices,

$$A^T = A$$

defines a symmetric matrix, while

$$A^T = -A$$

defines a skew-symmetric matrix.

For complex matrices, the transpose is replaced by the conjugate transpose.

Real Matrices	**Complex Matrices**
$A^T = A$	$A^* = A$
Symmetric	Hermitian
$A^T = -A$	$A^* = -A$
Skew-symmetric	Skew-Hermitian

Thus, Hermitian matrices are the natural complex generalization of symmetric matrices.

7 Properties of Hermitian Matrices

Let A be a Hermitian matrix.

7.1 Property 1: Diagonal Elements are Real

$$\boxed{a_{ii} \in \mathbb{R}}.$$

7.2 Property 2: Eigenvalues are Real

If

$$Ax = \lambda x, \qquad x \neq 0,$$

then

$$\boxed{\lambda \in \mathbb{R}}.$$

Proof

Since $A = A^*$,

$$x^* Ax = x^*(\lambda x) = \lambda x^* x.$$

For a Hermitian matrix, $x^* Ax$ is real. Also,

$$x^* x > 0.$$

Therefore, λ must be real.
Hence, every eigenvalue of a Hermitian matrix is real.

7.3 Property 3: Orthogonality of Eigenvectors

If

$$Ax = \lambda x, \qquad Ay = \mu y,$$

where

$$\lambda \neq \mu,$$

then

$$\boxed{x^* y = 0}.$$

Thus, eigenvectors corresponding to distinct eigenvalues of a Hermitian matrix are orthogonal.

7.4 Property 4: Determinant is Real

Since all eigenvalues of a Hermitian matrix are real,

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

is real.
Hence,

$$\boxed{\det(A) \in \mathbb{R}}.$$

8 Properties of Skew-Hermitian Matrices

Let A be skew-Hermitian:

$$A^* = -A.$$

8.1 Property 1: Diagonal Elements are Purely Imaginary

$$\boxed{a_{ii} = ib_i, \qquad b_i \in \mathbb{R}}.$$

8.2 Property 2: Eigenvalues are Purely Imaginary

If

$$Ax = \lambda x,$$

then

$$\boxed{\lambda = i\alpha, \qquad \alpha \in \mathbb{R}}.$$

Proof

We have

$$x^* A x = \lambda x^* x.$$

Since

$$A^* = -A,$$

we get

$$\overline{x^* A x} = x^* A^* x = -x^* A x.$$

Therefore, $x^* A x$ is purely imaginary. Since $x^* x$ is positive real, λ must be purely imaginary.

8.3 Property 3: Determinant

If A is an $n \times n$ skew-Hermitian matrix, its eigenvalues have the form

$$i\alpha_1, i\alpha_2, \ldots, i\alpha_n,$$

where $\alpha_j \in \mathbb{R}$.

Therefore,

$$\det(A) = i^n \alpha_1 \alpha_2 \cdots \alpha_n.$$

Thus,

$$\boxed{\det(A) = i^n r, \qquad r \in \mathbb{R}}.$$

9 Hermitian and Skew-Hermitian Decomposition

Every complex square matrix can be uniquely expressed as the sum of a Hermitian matrix and a skew-Hermitian matrix.

Let A be any complex square matrix. Define

$$\boxed{H = \frac{A + A^*}{2}}$$

and

$$\boxed{K = \frac{A - A^*}{2}.}$$

Then

$$\boxed{A = H + K}.$$

Now,

$$H^* = \left(\frac{A + A^*}{2}\right)^* = \frac{A^* + A}{2} = H.$$

Therefore, H is Hermitian.
Similarly,

$$K^* = \left(\frac{A - A^*}{2}\right)^* = \frac{A^* - A}{2} = -K.$$

Therefore, K is skew-Hermitian.
Hence,

$$\boxed{A = \underbrace{\frac{A + A^*}{2}}_{\text{Hermitian}} + \underbrace{\frac{A - A^*}{2}}_{\text{Skew-Hermitian}}}.$$

This decomposition is unique.

10 Example of Decomposition

Let

$$A = \begin{pmatrix} 1+i & 2+3i \\ 4-i & 5 \end{pmatrix}.$$

First,

$$A^* = \begin{pmatrix} 1-i & 4+i \\ 2-3i & 5 \end{pmatrix}.$$

The Hermitian part is

$$H = \frac{A + A^*}{2}.$$

Therefore,

$$\boxed{H = \begin{pmatrix} 1 & 3 + \frac{3i}{2} \\ 3 - \frac{3i}{2} & 5 \end{pmatrix}}.$$

The skew-Hermitian part is

$$K = \frac{A - A^*}{2}.$$

Thus,

$$\boxed{K = \begin{pmatrix} i & -1 + \frac{3i}{2} \\ 1 + \frac{3i}{2} & 0 \end{pmatrix}}.$$

Hence,

$$\boxed{A = H + K}.$$

11 Important Theorems

Theorem 11.1. *If A and B are Hermitian matrices, then $A+B$ is Hermitian. Moreover, if $c \in \mathbb{R}$, then cA is Hermitian.*

Proof. Since $A^* = A$ and $B^* = B$,

$$(A+B)^* = A^* + B^* = A + B.$$

Hence, $A+B$ is Hermitian.

Also,

$$(cA)^* = \overline{c}A^* = cA,$$

because $c \in \mathbb{R}$. □

Theorem 11.2. *If A and B are skew-Hermitian matrices, then $A+B$ is skew-Hermitian.*

Proof. Since $A^* = -A$ and $B^* = -B$,

$$(A+B)^* = A^* + B^* = -A - B = -(A+B).$$

Therefore, $A+B$ is skew-Hermitian. □

12 Relation Between Hermitian and Skew-Hermitian Matrices

If A is Hermitian, then

$$iA$$

is skew-Hermitian.

Indeed,

$$(iA)^* = \bar{i}A^* = -iA = -(iA).$$

Therefore,

$$\boxed{A \text{ Hermitian} \implies iA \text{ skew-Hermitian.}}$$

Similarly, if A is skew-Hermitian, then

$$-iA$$

is Hermitian.

Thus,

$$\boxed{A \text{ skew-Hermitian} \implies -iA \text{ Hermitian.}}$$

13 Product of Hermitian Matrices

If A and B are Hermitian matrices, then

$$(AB)^* = B^*A^* = BA.$$

Therefore, AB is Hermitian if and only if

$$AB = BA.$$

Hence,

$$\boxed{\text{The product of two Hermitian matrices is Hermitian if they commute.}}$$

14 Positive Definite Hermitian Matrix

A Hermitian matrix A is called **positive definite** if

$$\boxed{x^*Ax > 0}$$

for every non-zero vector x.

Similarly, A is called **positive semidefinite** if

$$\boxed{x^*Ax \geq 0}$$

for every vector x.

For a Hermitian matrix,

$$\boxed{A \text{ is positive definite} \iff \text{all eigenvalues of } A \text{ are positive.}}$$

15 Spectral Theorem for Hermitian Matrices

Theorem 15.1 (Spectral Theorem)**.** *Every Hermitian matrix A can be diagonalized by a unitary matrix. That is,*

$$\boxed{A = UDU^*},$$

where U is a unitary matrix and D is a diagonal matrix whose diagonal entries are real eigenvalues of A.

Thus, every Hermitian matrix is **unitarily diagonalizable**.

16 Unitary Matrix

A complex square matrix U is called a **unitary matrix** if

$$\boxed{U^*U = UU^* = I}.$$

Therefore,

$$\boxed{U^{-1} = U^*}.$$

Unitary matrices preserve the length of vectors:

$$\boxed{\|Ux\| = \|x\|}.$$

17 Comparison: Hermitian vs. Skew-Hermitian

Hermitian Matrix	**Skew-Hermitian Matrix**
$A^* = A$	$A^* = -A$
Diagonal entries are real	Diagonal entries are purely imaginary
Eigenvalues are real	Eigenvalues are purely imaginary
x^*Ax is real	x^*Ax is purely imaginary
$a_{ij} = \overline{a_{ji}}$	$a_{ij} = -\overline{a_{ji}}$
Generalization of symmetric matrices	Generalization of skew-symmetric matrices
Unitarily diagonalizable	Unitarily diagonalizable

18 Important Formulae for Examination

The following formulae are important:

$$\boxed{A^* = \overline{A}^T}$$

$$\boxed{A \text{ is Hermitian} \iff A^* = A}$$

$$\boxed{A \text{ is skew-Hermitian} \iff A^* = -A}$$

Hermitian part:

$$\boxed{H = \frac{A + A^*}{2}}$$

Skew-Hermitian part:

$$\boxed{K = \frac{A - A^*}{2}}$$

Decomposition:

$$\boxed{A = H + K}$$

For a Hermitian matrix:

$$\boxed{\lambda \in \mathbb{R}}.$$

For a skew-Hermitian matrix:

$$\boxed{\lambda \in i\mathbb{R}}.$$

19 Practice Questions

Short Answer Questions

1. Define a Hermitian matrix.
2. Define a skew-Hermitian matrix.
3. What is the conjugate transpose of a matrix?
4. What can you say about the diagonal elements of a Hermitian matrix?
5. What can you say about the eigenvalues of a Hermitian matrix?
6. What can you say about the eigenvalues of a skew-Hermitian matrix?
7. Show that iA is skew-Hermitian if A is Hermitian.
8. Show that every complex square matrix can be written as the sum of a Hermitian and a skew-Hermitian matrix.

Numerical Questions

1. Determine whether
$$A = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$$
is Hermitian.
2. Determine whether
$$A = \begin{pmatrix} 2i & 1+i \\ -1+i & -3i \end{pmatrix}$$
is skew-Hermitian.
3. Find the Hermitian and skew-Hermitian parts of
$$A = \begin{pmatrix} 1+i & 2 \\ 3i & 4-i \end{pmatrix}.$$
4. Find the eigenvalues of a given Hermitian matrix and verify that they are real.
5. Prove that eigenvectors corresponding to distinct eigenvalues of a Hermitian matrix are orthogonal.