1 Introduction

Change of variables is the multivariable generalization of single-variable u-substitution. It allows us to transform complicated regions of integration into simpler ones (such as rectangles or boxes) or simplify complex integrand functions.

2 Single-Variable Review

In single-variable calculus, $x = g(u)$ yields:

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

Here, $g'(u)$ acts as a local stretching factor converting intervals of u to intervals of x.

3 The Jacobian Determinant

In two dimensions, let T be a transformation given by $x = x(u, v)$ and $y = y(u, v)$. The scaling factor for area elements is given by the **Jacobian determinant**:

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

The area element converts as:

$$dA = dx\,dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv$$

4 Change of Variables Theorem (Double Integrals)

If $T : S \to R$ is a C^1, one-to-one transformation mapping a region S in the uv-plane onto R in the xy-plane, then:

$$\iint_R f(x, y)\,dx\,dy = \iint_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv$$

5 Standard Coordinate Transformations

5.1 Polar Coordinates

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r$$

$$dA = dx\,dy = r\,dr\,d\theta$$

5.2 Spherical Coordinates

$$x = \rho\sin(\varphi)\cos(\theta), \quad y = \rho\sin(\varphi)\sin(\theta), \quad z = \rho\cos(\varphi)$$

$$dV = dx\,dy\,dz = \rho^2\sin(\varphi)\,d\rho\,d\theta\,d\varphi$$