Cauchy-Euler Equations

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1 Overview

A **Cauchy-Euler equation** is a special type of differential equation with variable coefficients where the exponent of the independent variable x matches the degree of differentiation.

2 Standard Form

The general second-order homogeneous Cauchy-Euler equation is given by:

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0, \quad x > 0$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

3 Derivation of the Characteristic Equation

Assuming a solution of the form $y = x^m$, we calculate the required derivatives:

$$y' = mx^{m-1}$$
$$y'' = m(m-1)x^{m-2}$$

Substituting into the original differential equation yields:

$$ax^2\left(m(m-1)x^{m-2}\right) + bx\left(mx^{m-1}\right) + cx^m = 0$$

$$[am(m-1) + bm + c]\,x^m = 0$$

Since $x^m \neq 0$ for $x > 0$, we get the **auxiliary polynomial**:

$$am^2 + (b-a)m + c = 0$$

4 General Solution Cases

4.1 Case 1: Real and Distinct Roots ($m_1 \neq m_2$)

If the characteristic equation has two distinct real roots m_1 and m_2, the general solution is:

$$y(x) = C_1x^{m_1} + C_2x^{m_2}$$

4.2 Case 2: Repeated Real Roots ($m_1 = m_2 = m$)

If the roots are identical, reduction of order yields a second linearly independent solution $x^m \ln(x)$:

$$y(x) = C_1x^m + C_2x^m\ln(x)$$

4.3 Case 3: Complex Conjugate Roots ($m = \alpha \pm i\beta$)

Using $x^{\alpha \pm i\beta} = x^{\alpha} e^{\pm i\beta \ln(x)} = x^{\alpha} [\cos(\beta \ln(x)) \pm i \sin(\beta \ln(x))]$, the real-valued general solution is:

$$y(x) = x^{\alpha} [C_1 \cos(\beta \ln(x)) + C_2 \sin(\beta \ln(x))]$$

5 Transformation Method

Alternatively, substituting $x = e^t \implies t = \ln(x)$ converts the equation to a constant-coefficient ODE:

$$a\frac{d^2y}{dt^2} + (b - a)\frac{dy}{dt} + cy = 0$$