$$= \int_{y=0}^{2} \int_{x=0}^{2} \left\{2\left(4-x^2\right)x + y\left(4-x^2\right)\right\} dx dy$$

$$= \int_{y=0}^{2} \left(16-8+8y-\frac{8}{3}y\right) dy$$

$$= \left(8y+\frac{16}{3}\frac{y^2}{2}\right)_0^2 = 16+\frac{32}{3} = \frac{80}{3}$$

**Example 6 :** Evaluate the value of $\iiint_V div\ \vec{F}\, dV$, where $\vec{F} = 4x\hat{i} - 2\hat{j} + z^2\hat{k}$ and V is the region bounded by $x^2 + y^2 = 4$ ; $z = 0$ and $z = 3$.

**Sol.** First we take

$$div\vec{F} = \nabla.\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\right)$$

Now $\iiint_V div\vec{F} dV = \iint_R \int_{z=0}^{3} (4-4y+2z) dx\, dy\, dz$

$$= \iint_R \left[4z - 4yz + z^2\right]_0^3 dx\, dy$$

$$= \iint_R \left[12(1-y)+9\right] dx\, dy$$

(Taking parametric equation of the curve $x^2 + y^2 + 4$ i.e., $x = r\,cos\theta$, $y = r\,sin\theta$ $\Rightarrow$ $dx\,dy = r\,d\theta dr$)

$$= \iint_R (21 - 12r\,sin\theta) rdr\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} (21 - 12r\,sin\theta) rdr\, d\theta$$

$$= \int_0^{2\pi} \left[21\frac{r^2 2}{} - 4r^3\,sin\theta\right]_0^2 d\theta$$

$$= \int_0^{2\pi} (42 - 32\,sin\theta) d\theta$$

$$= (42\theta + 32\,cos\theta)_0^{2\pi}$$

$$= 84\pi + 32 - 32 - 84\pi$$
$$= 0$$

## 3.8 Self Learning Exercise

**Q.1** Work done by a particle in a force field $\vec{F}$ on moving particle from point A to point B is given by..

**Q.2** Write the unit normal vector to surface $x^2 + y^2 + z^2 = a^2$ at point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

**Section – B (Short Answer type Question)**

**Q.3** Find the circulation of $\vec{F} = \frac{-y}{x^2 + y^2}\hat{i} + \frac{x}{x^2 + y^2}\hat{j}$ round the circle $x^2 + y^2 = 1$ in $xy$ plane.

**Q.4** Evaluate $\iint_S \vec{F}.\hat{n}dS$ where $\vec{F} = \left(x^2 + y^2\right)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ & $S$ is surface of plane $2x + y + 2z = 6$ in first octant.

**Section – C (Long Answer type Question)**

**Q.5** Evaluate $\iiint_V diV\vec{F}dV$ where $\vec{F} = \left(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right)$ & $V$ is volume bounded by sphere $x^2 + y^2 + z^2 = 1$ above $xy$-plane.

**Q.6** Evaluate $\iint_S Curl\,\vec{F}ds$ where $\vec{F} = xy\hat{i} - 2yz\hat{j} - zx\hat{k}$ & $S$ is open surface of rectangle parallel piped formed by planes $x = 0$, $x = 1$, $y = 0$, $y = 2$ & $z = 3$ above $xy$-plane.

## 3.9 Gauss Divergence Theorem

The Gauss divergence theorem transforms double (surface) integral into volume Integral with the help of divergence of a vector point function. Gauss's divergence theorem is also known as **Ostogradsky's theorem**.

**Statement :** If $\vec{F}$ be a continuously differential vector point function and $S$ is a closed, smooth and orientable surface enclosing a region $V$, then

$$\boxed{\int_S \vec{F}.\hat{n}dS = \int_S \text{div}\,\vec{F}.dV = \int_V \left(\vec{\nabla}.\vec{F}\right).dV}$$

or $$\boxed{\iint_S \vec{F}.\hat{n}\,dxdy = \iiint_V \text{div}\,\vec{F}\,dxdydz},$$

where $\hat{n}$ is the unit outward drawn normal vector on the surface $S$.

## 3.10 Applications of Gauss's divergence theorem

The divergence theorem finds applications in evaluating the integrals of dot and cross products of vector fields and scalar fields.

**(A) Product of a scalar function $g(x,y,z)$ and a vector field $\vec{F}(x,y,z)$**

The surface integral, with respect to a surface $S$, of the scalar product $g\vec{F}$ is evaluated by using the following result:

$$\oiint_S g\,\vec{F}.\hat{n}ds = \iiint_V \left[\vec{F}.\left(\vec{\nabla}g\right) + g\left(\vec{\nabla}.\vec{F}\right)\right]dV$$

**(B) Cross product of two vector fields $\vec{F}\times\vec{G}$ :**

The surface integral, with respect to a surface $S$, of the cross product $\vec{F}\times\vec{G}$ is evaluated by using the following result :

$$\oiint_S \left(\vec{F}\times\vec{G}\right).dS = \iiint_V \left[\vec{G}.\left(\vec{\nabla}\times\vec{F}\right) - \vec{F}.\left(\vec{\nabla}\times\vec{G}\right)\right]dV$$

**(C) Product of scalar function $f(x,y,z)$ and a non zero constant vector**

Following result exists for the evaluation of surface integral of product of a scalar function, $f$, and a non zero constant vector.

$$\oiint_S f dS = \iiint_V \vec{\nabla}f\,dV$$

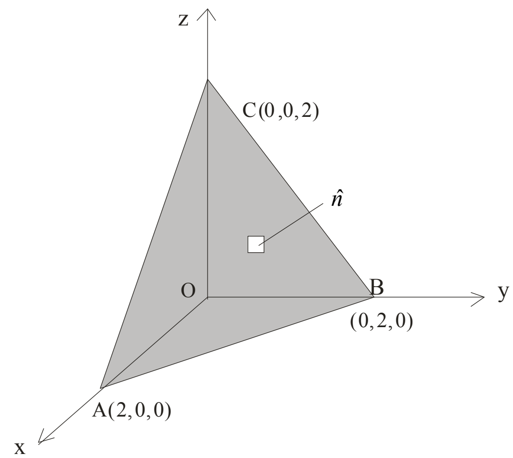
**(D) Cross product of a vector field $\vec{F}$ and a non-zero constant vector**

Application of divergence theorem to the cross product of a vector field $\vec{F}$ and a non-zero constant vector, gives following result:

$$\oiint_S d\vec{S}\times\vec{F} = \iiint_V \left(\vec{\nabla}\times\vec{F}\right)dV$$

**Example 7 :** Evaluate the surface integral $\iint\limits_S \vec{F}.\hat{n}ds$, where $\vec{F} = \left(x^2 + y^2 + z^2\right)\left(\hat{i} + \hat{j} + \hat{k}\right)$, $S$ is the tetrahedron $x = 0$, $y = 0$, $z = 0$, $x + y + z = 2$ and $\hat{n}$ is the unit outward drawn normal to the closed surface $S$.

**Sol.** It is convenient to use Gauss's theorem for the evaluation of the integral.




By Gauss's theorem

$$\iint\limits_S \vec{F}.\hat{n}ds = \iiint\limits_V div\,\vec{F}.dV$$

Here $\quad \vec{F} = \left(x^2 + y^2 + z^2\right)\left(\hat{i} + \hat{j} + \hat{k}\right)$

$$\therefore \qquad div\,\vec{F} = \sum \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right) = 2x + 2y + 2z$$

$$\iint\limits_S \vec{F}.\hat{n}ds = \iiint\limits_V 2\left(x + y + z\right)dV$$

$$= \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2\left(x + y + z\right)\,dx\,dy\,dz$$

$$= 2\int_0^2 \int_0^{2-x} \left[\left(x + y\right)z + \frac{z^2}{2}\right]_0^{2-x-y} dx\,dy$$

$$= 2\int_0^2 \int_0^{2-x} \left[\left(x + y\right)\left(2 - x - y\right) + \frac{1}{2}\left(2 - x - y\right)^2\right] dx\,dy$$

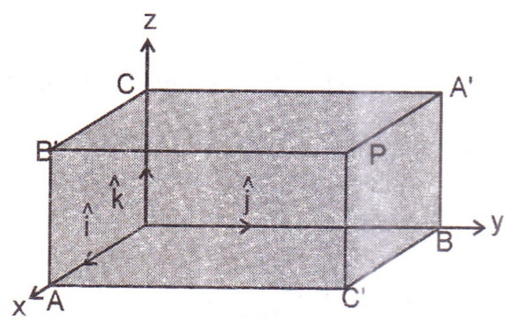
$$\iint\limits_S \vec{F}.\hat{n}ds = 2\int_0^2 \int_0^{2-x} \left[2 - \frac{\left(x + y\right)^2}{2}\right] dx\,dy$$

$$= 2\int_0^2 \left[2y - \frac{(x+y)^3}{6}\right]_0^{2-x} dx$$

$$= 2\int_0^2 \left[2(2-x) - \frac{8}{6} + \frac{x^3}{6}\right] dx \quad = 2\left(\frac{8}{3}x - x^2 + \frac{x^4}{24}\right)_0^2 = 4$$

**Example 8 :** Verify divergence theorem for $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ taken over the rectangular parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

**Sol.**




For verification of divergence theorem, we shall evaluate the volume and surface integrals separately and show that they are equal

Now, div $\vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy) = 2(x+y+z)$

$$\therefore \qquad \iiint_V div\,\vec{F} dV = \int_0^c \int_0^b \int_0^a 2(x+y+z)\,dx\,dy\,dz$$

$$= \int_0^c \int_0^b 2\left[\left(\frac{x^2}{2} + yx + zx\right)\right]_0^a dydz$$

$$= \int_0^c \int_0^b 2\left(\frac{a^2}{2} + ya + za\right) dydz = \int_0^c 2\left[\left(\frac{a^2}{2}y + \frac{ay^2}{2} + azy\right)\right]_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right) dz = 2\left[\frac{a^2 b}{2} z + \frac{ab^2}{2} z + ab\frac{z^2}{2}\right]_0^c$$

$$= a^2bc + ab^2c + abc^2 = abc(a+b+c) \qquad \ldots (1)$$

To evaluate the surface integrals, divide the closed surface $S$ of the rectangular parallelepiped into 6 parts.

$S_1$ : the face $OAC'B$, $S_2$ : the face $OBA'C$, $S_3$ : the face $AC'PB'$, $S_4$ : the face $AC'PB'$, $S_5$ : the face $OCB'A$, $S_6$ : the face $BA'PC'$

Also

$$\iint_S \vec{F}.\hat{n}dS = \iint_{S_1} \vec{F}.\hat{n}dS + \iint_{S_2} \vec{F}.\hat{n}dS + \iint_{S_3} \vec{F}.\hat{n}dS$$
$$+ \iint_{S_4} \vec{F}.\hat{n}dS + \iint_{S_5} \vec{F}.\hat{n}dS + \iint_{S_6} \vec{F}.\hat{n}dS$$

On $S_1(z=0)$, we have $\hat{n} = -\hat{k}$, $\vec{F} = x^2\hat{i} + y^2\hat{j} - xy\hat{k}$

So that $\quad \vec{F}.\hat{n} = \left(x^2\hat{i} + y^2\hat{j} - xy\hat{k}\right).\left(-\hat{k}\right) = xy$

$$\therefore \quad \iint_S \vec{F}.\hat{n}dS = \int_0^b\int_0^a xydxdy = \int_0^b \left[ y\frac{x^2}{2} \right] dy = \frac{a^2}{2}\int_0^b ydy = \frac{a^2b^2}{4}$$

On $S_2(z=c)$, we have $\hat{n} = \hat{k}$, $\vec{F} = \left(x^2 - cy\right)\hat{i} + \left(y^2 - cy\right)\hat{j} + \left(c^2 - xy\right)\hat{k}$

So that $\quad \vec{F}.\hat{n} = \left[\left(x^2 - cy\right)\hat{i} + \left(y^2 - cy\right)\hat{j} + \left(c^2 - xy\right)\hat{k}\right].\hat{k} = c^2 - xy$

$$\iint_{S_2} \vec{F}.\hat{n}dS = \int_0^b\int_0^a \left(c^2 - xy\right)dx\,dy = \int_0^b \left[ c^2a - \frac{a^2}{2}y \right] dy = abc^2 - \frac{a^2b^2}{4}$$

On $S_3(x=0)$, we have $\hat{n} = -\hat{i}$, $\vec{F} = -yz\hat{i} + y^2\hat{j} + z^2\hat{k}$

So that $\quad \vec{F}.\hat{n} = \left(-yz\hat{i} + y^2\hat{j} + z^2\hat{k}\right).\left(-\hat{i}\right) = yz$

$$\therefore \quad \iint_{S_3} \vec{F}.\hat{n}dS = \int_0^c\int_0^b yz\,dy\,dz = \int_0^c \frac{b^2}{2}zdz = \frac{b^2c^2}{4}$$

On $S_4(x=a)$, we have $\hat{n} = \hat{i}$, $\vec{F} = \left(a^2 - yz\right)\hat{i} + \left(y^2 - az\right)\hat{j} + \left(z^2 - ay\right)\hat{k}$

So that $\quad \vec{F}.\hat{n} = \left[\left(a^2 - yz\right)\hat{i} + \left(y^2 - az\right)\hat{j} + \left(z^2 - ay\right)\hat{k}\right].\,\hat{i} = a^2 - yz$

$$\therefore \quad \iint_{S_4} \vec{F}.\hat{n}dS = \int_0^c\int_0^b \left(a^2 - yz\right)dydz = \int_0^c \left( a^2b - \frac{b^2}{2}z \right) dz = a^2bc - \frac{b^2c^2}{4}$$

On $S_5(y=0)$, we have $\hat{n} = -\hat{j}$, $\vec{F} = x^2\hat{i} - zx\hat{j} + z^2\hat{k}$

So that $\quad \vec{F}.\hat{n} = \left(x^2\hat{i} - zx\hat{j} + z^2\hat{k}\right).\left(-\hat{j}\right) = zx$

$$\therefore \quad \iint_{S_5} \vec{F}.\hat{n}dS = \int_0^a\int_0^c zx\,dz\,dx = \int_0^a \frac{c^2}{2}x\,dx = \frac{a^2c^2}{4}$$

On $S_6(y=b)$, we have $\hat{n} = \hat{j}$, $\vec{F} = \left(x^2 - bz\right)\hat{i} + \left(b^2 - zx\right)\hat{j} + \left(z^2 - bx\right)\hat{k}$

So that $\quad \vec{F}.\hat{n} = \left[\left(x^2 - bz\right)\hat{i} + \left(b^2 - zx\right)\hat{j} + \left(z^2 - bx\right)\hat{k}\right].\hat{j} = b^2 - zx$

$$\therefore \quad \iint_{S_6} \vec{F}.\hat{n}dS = \int_0^a \int_0^c \left(b^2 - zx\right) dz\, dx$$

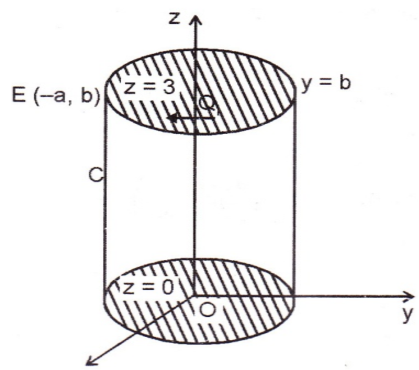
$$= \int_0^c \left(b^2 c - \frac{c^2}{2} x\right) dx = ab^2 c - \frac{a^2 c^2}{4}$$

$$\therefore \quad \iint_S \vec{F}.\hat{n}dS = \frac{a^2 b^2}{4} + abc^2 + \frac{a^2 b^2}{4} + \frac{b^2 c^2}{4} + a^2 bc - \frac{b^2 c^2}{4} + \frac{a^2 c^2}{4} + ab^2 c - \frac{a^2 c^2}{4} \qquad \ldots (2)$$

The equality of (1) and (2) verifies divergence theorem.

**Example 9 :** Verify divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by the cylinder $x^2 + y^2 = 4$, $z = 0$, $z = 3$.

**Sol.** Since $\quad$ div $\vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}\left(-2y^2\right) + \frac{\partial}{\partial z}\left(z^2\right) = 4 - 4y + 2z$




$$\therefore \quad \iiint_V div\, \vec{F} dV = \iiint_V (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^3 (4 - 4y + 2z)\, dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4y + z^2\right]_0^3 dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9)\, dy\, dx = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21\, dy\, dx$$

[since $12y$ is an old function $\therefore \int_{-a}^{a} 12y\, dy = 0$]

$$= \int_{-2}^{2} 42\sqrt{4-x^2}\,dx = 84\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 84\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 = 84\left[2\sin^{-1}1\right]$$

$$= 84\left[2\times\frac{\pi}{2}\right] = 84\pi \qquad \ldots (1)$$

To evaluate the surface integral divide the closed surface $S$ of the cylinder into 3 parts.

$S_1$ : the circular base in the plane $z=0$

$S_2$ : the circular top in the plane $z=3$

$S_3$ : the curved surface of the cylinder, given by the equation $x^2+y^2=4$

Also $\qquad \iint_S \vec{F}.\hat{n}dS = \iint_{S_1} \vec{F}.\hat{n}dS + \iint_{S_2} \vec{F}.\hat{n}dS + \iint_{S_3} \vec{F}.\hat{n}dS$

On $S_1(z=0)$, we have $\hat{n} = -\hat{k}$, $\vec{F} = 4x\hat{i} - 2y^2\hat{j}$

So that $\qquad \vec{F}.\hat{n} = \left(4x\hat{i} - 2y^2\hat{j}\right).\left(-\hat{k}\right) = 0$

$\therefore \qquad \iint_{S_1} \vec{F}.\hat{n}dS = 0$

On $S_2(z=3)$, we have $\hat{n} = \hat{k}$, $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + 9\hat{k}$

So that $\qquad \vec{F}.\hat{n} = \left(4x\hat{i} - 2y^2\hat{j} + 9\hat{k}\right).\hat{k} = 9$

$\therefore \qquad \iint_{S_2} \vec{F}.\hat{n}dS = \iint_{S_2} 9\,dx\,dy = 9\iint_{S_2} dx\,dy$

$= 9\times$ area of surface $S_2 = 9\left(\pi.2^2\right) = 36\pi$

On $\qquad S_3$, $x^2+y^2=4$

A vector normal to the surface $S_3$ is given by $\nabla\left(x^2+y^2-4\right) = 2x\hat{i} + 2y\hat{j}$

$\therefore \qquad \hat{n}$ = a unit vector normal to surface $S_3$

$$= \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4\times 4}}, \qquad [\text{since } x^2+y^2=4]$$

$$= \frac{x\hat{i}+y\hat{j}}{2}$$

$$\vec{F}.\hat{n} \quad = \left(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\right).\left(\frac{x\hat{i} + y\hat{j}}{2}\right) = 2x^2 - y^3$$

Also, on $S_3$, i.e., $x^2 + y^2 = 4$, $x = 2\cos\theta$, $y = \sin\theta$ and $dS = 2d\theta\, dz$.

To cover the whole surface $S_3$, $z$ varies from 0 to 3 and $\theta$ varies from 0 to $2\pi$.

$$\therefore \qquad \iint_{S_3} \vec{F}.\hat{n}dS = \int_0^{2\pi}\int_0^3 \left[2\left(2\cos\theta\right)^2 - \left(2\sin\theta\right)^3\right]2dz\,d\theta$$

$$= \int_0^{2\pi} 16\left(\cos^2\theta - \sin^3\theta\right)\times 3d\theta$$

$$= 48\int_0^{2\pi}\left(\cos^2\theta - \sin^3\theta\right) = d\theta = 48\pi$$

$$\left(\text{since } \int_0^{2\pi}\cos^2\theta\,d\theta = \int_0^{\pi/2}\cos^2 d\theta = 4\times\frac{1}{2}\times\frac{\pi}{2} = \pi, \int_0^{2\pi}\sin^3\theta d\theta = 0\right)$$

$$\therefore \qquad \iint_S \vec{F}.\hat{n}dS = 0 + 36\pi + 48\pi = 84\pi$$

The equality of (1) and (2) verifies divergence theorem.

## 3.11 Stoke's Theorem

The Stoke's theorem transforms line integral into surface integral with the help of curl of a vector point function. Stoke's theorem is the vector form of Green's theorem or generalized Green's theorem.

**Statement :** If $S$ is the open surface bounded by a closed curve $C$ and $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ is any continuously differentiable vector function, then

$$\boxed{\int_C \vec{F}.d\vec{r} = \int_S Curl\,\vec{F}.\hat{n}\,ds = \int\left(\nabla\times\vec{F}\right).\hat{n}ds},$$

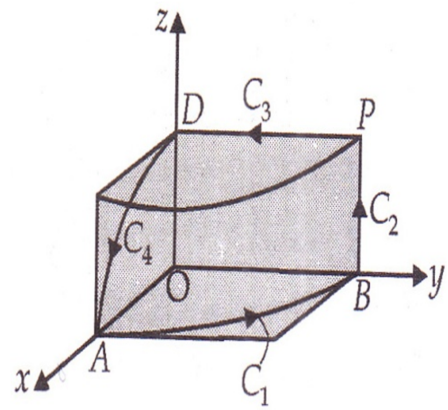
Where $\hat{n}$ is the unit outward normal drawn to the surfaces.

**Example 10 :** Evaluate $\iint_S \left(\nabla\times\vec{A}\right).\hat{n}\,ds$ over the surface of intersection of cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$, which is included in the first octant given that $\vec{A} = 2yz\hat{i} - \left(x - 3y - 2\right)\hat{j} + \left(x^2 + z\right)\hat{k}$.

**Sol.** By Stoke's theorem

$$\iint_S \left(\nabla\times\vec{A}\right).\hat{n}\,ds = \int_C \vec{A}.d\vec{r}$$

Here $C$ is the curve consisting of four arcs namely $C_1 : AB$, $C_2 : BP$, $C_3 : PD$, $C_4 : DA$. Thus, we evaluate RHS of (1), along these four arcs one by one.




Along $C_1$ : $z = 0$, $x^2 + y^2 = a^2$ $y$ varies from 0 to $a$.

$$\int_{C_1} \vec{A}.d\vec{r} = \int_{C_1} \left[2yzdx - (x+3y-2)dy + (x^2+z)dz\right]$$

$$= -\int_{C_1} (x+3y-2)dy$$

$$= -\int_0^a \left[\sqrt{a^2-y^2} + 3y - 2\right]dy$$

$$= -\left[\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2} sin^{-1}\frac{y}{2} + \frac{3}{2}y^2 - 2y\right]_0^a$$

$$\int_{C_1} \vec{A}.d\vec{r} = -\frac{\pi a^2}{2} - \frac{3a^2}{2} + 2a \qquad (2)$$

Along $C_2$ : $x = 0$, $y = a$; $dx = 0$; $dy = 0$ and $z$ varies from 0 to $a$

$$\therefore \quad \int_{C_2} \vec{A}.d\vec{r} = \int_{C_2} zdz = \int_0^a zdz = \left(\frac{z^2}{2}\right)_0^a = \frac{a^2}{2} \qquad \dots (3)$$

Along $C_3$ : $x = 0$, $z = a$; $dx = 0$; $dz = 0$ and $y$ varies from $a$ to 0

$$\therefore \quad \int_{C_3} \vec{A}.d\vec{r} = \int_{C_3} (3y-2)dy = -\int_0^a (3y-2)dy$$

$$= \left[-\frac{3y^2}{2} + 2y\right]_a^0 = \frac{3a^2}{2} - 2a \qquad \dots (4)$$

Along $C_4$ : $y=0$, $x^2+z^2=a^2$, $z$ varies from $a$ to 0

$$\therefore \quad \int_{C_4} \vec{A}.d\vec{r} = \int_a^0 \left(x^2+z\right)dz = \int_a^0 \left(a^2-z^2+z\right)dz$$

$$= \left(a^2 z - \frac{1}{3}z^3 + \frac{z^2}{2}\right)_a^0 = -\frac{2a^3}{3} - \frac{a^2}{2} \qquad \ldots (5)$$

Thus, the desired integral is sum of (2), (3), (4), (5)

$$\text{i.e.} \quad \iint_S \left(\nabla \times \vec{A}\right).\hat{n}\,ds = -\frac{\pi a^2}{4} - \frac{3a^2}{2} + 2a + \frac{a^2}{2} + \frac{3a^2}{2} - 2a - \frac{2a^3}{3} - \frac{a^2}{2}$$

$$= -\frac{\pi a^2}{4} - \frac{2a^3}{3} = -\frac{a^2}{12}\left(3\pi + 8a\right) \qquad \text{Ans.}$$

**Example 11 :** Verify Stoke's theorem for $F = \left(x^2+y-5\right)i + 3xyj + \left(2xz+z^3\right)k$ over the surface of the hemisphere $x^2+y^2+z^2=16$ above the $xy$-plane.

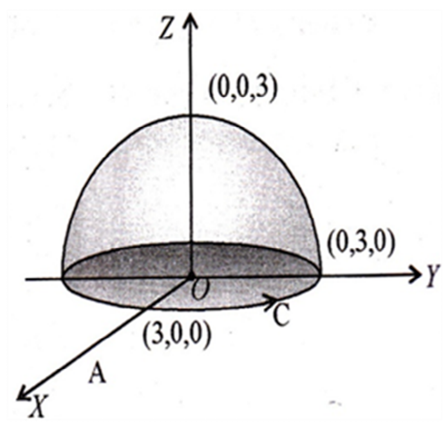
**Sol.** Here $S$ is the surface $x^2+y^2+z^2=16$ and $C$ is the boundary of the hemispherical surface and is given by $C$ : $x^2+y^2=16$.

$\therefore$ $x=4\cos t$, $y=4\sin t$, $z=0$, $0 \le t \le 2\pi$

$\Rightarrow$ $r = 4\cos ti + 4\sin tj + 0k$

$dr = \left(-4\sin ti + 4\cos tj\right)dt$

and $\quad F = \left(16\cos^2 t + 4\sin t - 4\right)i + 48\sin t.\cos tj$




Stoke's theorem is

$$\oint_C F.dr = \iint_R (\nabla \times F).\hat{n}\frac{dA}{|n.k|}$$

Where $R$ is the region in $xy$-plane bounded by curve $C$

L.H.S. of Stoke's theorem

$$= \oint_C F.dr$$

$$= \int_0^{2\pi}\left[\left(-64\sin t\cos^2 t + 16\sin^2 t - 16\sin t\right) + 192\sin t\cos^2 t\right]dt$$

$$= \int_0^{2\pi}\left\{128\cos^2 t\sin t - \frac{16(1-\cos t)}{2} + 16\sin t\right\}dt$$

$$= \left[\frac{-128\left(\cos^3 t\right)}{3} - 8t + 4\sin 2t - 16\cos t\right]_0^{2\pi} = -16\pi \qquad \ldots (1)$$

Now $\nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y - 4 & 3xy & 2xz + z^2 \end{vmatrix}$

$$= i(0-0) - j(2z-0) + k(3y-1)$$

$$= -2zj + (3y-1)k$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2xi + 2yj + 2zk}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{(xi + yj + zk)}{4}$$

$$= \frac{1}{4}\left[-2yz + (3y-1)z\right]$$

$$= \frac{yz - z}{4} = \frac{3}{4}(y-1)$$

R.H.S. of Stokes theorem $= \iint_R (\nabla \times F).\hat{n}\dfrac{dxdy}{|n.k|}$

$$= \iint_R \frac{z}{4}(y-1).\frac{dxdy}{z/4}$$

$$= \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (y-1)\,dydx$$

$$= \int_{-4}^{4} \left(y^2 - y\right)_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} dx$$

$$= \int_{-4}^{4} -2\sqrt{16-x^2}\, dx$$

$$= -2\left(\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2} sin^{-1}\frac{x}{4}\right)_{-4}^{4}$$

$$= 16\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$= -16\pi =$ L.H.S.    Hence verified.

**Example 12** If $F = \left(y^2 + z^2 + x^2\right)i + \left(z^2 + x^2 - y^2\right)j + \left(x^2 + y^2 - z^2\right)k$ Evaluate $\iint_S (\nabla \times F).\hat{n}ds$ taken over the surface $S = x^2 + y^2 - 2ax + az = 0$, $z = 0$

**Sol.** The given surface $S = x^2 + y^2 - 2ax + az = 0$ is bounded by the curve $C$ : $x^2 + y^2 - 2ax = 0$, $z = 0$.

or $\quad (x-a)^2 + y^2 = a^2$, $z = 0$

or $\quad x = a + a\,cos\,t$, $y = a\,sin\,t$, $z = 0$

$$\therefore r = a(1 + cos\,t)i + a\,sin\,tj$$

$$\Rightarrow dr = (-a\,sin\,tj + a\,cos\,tj)dt$$

and

$$F = \left[a^2 sin^2 t + a^2(1 + cos\,t)^3\right]i + \left(a^2(1 + cos\,t)^2 - a^2 sin^2 t\right)j + 2a^2(1 + cos\,t)k$$

By Stokes theorem

$$\iint_S (\nabla \times F).nds = \oint F.dr$$

$$= \int_0^{2\pi}\left[a^2\left\{sin^2 t + (1 + cos\,t)^2\right\}\left(-a\,sin\,t + a^2\left\{(1 + cos\,t)^2 - sin^2 t\right\}(a\,cos\,t)\right]dt$$

$$= a^3 \int_0^{2\pi}\left[-2\,sin\,t(1 + cos\,t)^2 + 2\left(1 - sin^2 t + cos\,t\right)cos\,t\right]dt$$

$$= 2a^3\left[\int_0^{2\pi} -\left(sin\,t + \frac{sin\,2t}{2}\right)dt + \int_0^{2\pi}\left(cos\,t + \frac{(1 + cos\,t)}{2} - 2\,sin^2 t\,cos\,t\right)dt\right]$$

$$= 2a^3\left[\int_0^{2\pi} -\left(\sin t + \frac{\sin 2t}{2}\right)dt + \int_0^{2\pi}\left(\cos t + \frac{(1+\cos t)}{2} - 2\sin^2 t\cos t\right)dt\right]$$

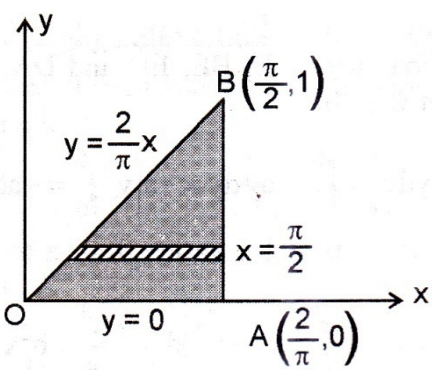
$$= 2a^2\left(\cos t + \frac{\cos 2t}{4}\right)_0^{2\pi} + \left[2a^3 \sin t + a^3\left(t + \frac{\sin 2t}{2}\right) - 2a^3\,\frac{\sin^3 t}{3}\right]_0^{2\pi}$$

$$= 2\pi a^3$$

**Example 13 :** Use Stoke's theorem to evaluate $\int_C \vec{F}.d\,\vec{r}$ where $\vec{F} = (\sin x - y)\hat{i} - \cos x\,\hat{j}$ and $C$ is the boundary of the triangle whose vertices are $(0,0)$, $(\pi/2,\,0)$ and $(\pi/2,\,1)$.

**Sol.** Evaluating $\oint_C \vec{F}.d\,\vec{r}$ by using Stoke's theorem means expressing the line integral in terms of its equivalent surface integral and then evaluating the surface integral.

By Stoke's theorem $\oint_C \vec{F}.d\,\vec{r} = \iint_S crul\,\vec{F}.d\,\vec{S}$, where $S$ is any open two-sided surface bounded by $C$.




To simplify the work, we shall choose $S$ as the plane surface $R$ in the $XY$-plane bounded by $C$.

$\therefore \oint_C \vec{F}.\vec{r} = \iint_S curel\,\vec{F}.\hat{k}\,dx\,dy$ [$\because$ for the $XOY$-plane, $\hat{n} = \hat{k}$ and $dS = dx\,dy$ ]

For this problem,

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\sin x - y0) & -\cos x & 0 \end{vmatrix} = (\sin x + 1)\hat{k}$$

$\therefore$ The given line integral

$$= \iint_R (1+\sin x)\,dx\,dy$$

$$= \int_0^1 \int_{\pi y/2}^{\pi/2} (1+\sin x)\,dx\,dy = \int_0^1 [x-\cos x]_{\pi y/2}^{\pi/2}$$

$$= \int_0^1 \left(\frac{\pi}{2} - \frac{\pi y}{2} + \cos\frac{\pi y}{2}\right) dy$$

$$= \left[\frac{\pi}{2} y - \frac{\pi y^2}{4} + \frac{2}{\pi} \sin\frac{\pi y}{2}\right]_0^1 = \frac{\pi}{4} + \frac{2}{\pi}$$

## 3.12 Summary

In this unit line integral in vector calculus is discussed. After that volume & surface integral are discussed. Conversion of line integral into surface integral & surface integral into volume integral is explained. Use of Gauss divergence theorem & Stokes theorem is discussed by using various examples.

## 3.13 Glossary

***Vector :*** A physical quantity having both magnitude & directions.

***Line Integral :*** An integral calculated along a curve.

***Surface Integral :*** An integral calculated on the surface of any curve.

## 3.14 Answer to Self Learning Exercise

**Ans.1 :** $\int_A^B \vec{F}.d\vec{r}$

**Ans.2 :** $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

**Ans.3 :** $2\pi$

**Ans.4 :** – 37

**Ans.5 :** $\frac{\pi}{12}$

**Ans.6 :** – 1

## 3.15 Exercise

**Section – A (Very Short Answer Type Question)**

**Q.1** State Gauss divergence theorem.

**Q.2** State Stoke’s theorem.