Adjoint and Inverse of a Matrix

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1 Minors and Cofactors

Let $A = [a_{ij}]$ be an $n \times n$ square matrix.

1.1 Definitions

- **Minor (M_{ij}):** The determinant of the $(n-1)\times(n-1)$ submatrix obtained by deleting row i and column j of matrix A.
- **Cofactor (C_{ij}):** The signed minor given by:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

2 Adjoint of a Matrix

The **adjoint** (or **adjugate**) of A, denoted $\text{adj}(A)$, is the transpose of its cofactor matrix C:

$$\text{adj}(A) = C^T \quad \Longrightarrow \quad [\text{adj}(A)]_{ij} = C_{ji}$$

2.1 2x2 General Case

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$\text{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

3 Inverse of a Matrix

A matrix A is non-singular (invertible) if and only if $\det(A) \neq 0$.

3.1 Fundamental Theorem

$$A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = \det(A) I_n$$

3.2 Inversion Formula

If $\det(A) \neq 0$, the inverse of A is given by:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

4 Worked Example

Find $\operatorname{adj}(A)$ and A^{-1} for:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}$$

Step 1: Compute Determinant

$$\det(A) = 1(0-24) - 2(0-20) + 3(0-5) = -24 + 40 - 15 = 1$$

Since $\det(A) = 1 \neq 0$, the matrix A is invertible.

Step 2: Compute Cofactors

$$\begin{aligned}
&C_{11} = +(-24) = -24, && C_{12} = -(-20) = 20, && C_{13} = +(-5) = -5 \\
&C_{21} = -(-18) = 18, && C_{22} = +(-15) = -15, && C_{23} = -(6-10) = 4 \\
&C_{31} = +(8-3) = 5, && C_{32} = -(4-0) = -4, && C_{33} = +(1-0) = 1
\end{aligned}$$

Step 3: Transpose Cofactor Matrix for Adjoint

$$\operatorname{adj}(A) = C^T = \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}$$

Step 4: Compute Inverse

$$A^{-1} = \frac{1}{1} \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}$$