De Morgan's and Bertrand's Tests

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1 Introduction

When standard higher-order tests such as D'Alembert's Ratio Test and Raabe's Test yield a limit of 1, they fail to classify the series. De Morgan and Bertrand's tests provide refined logarithmic criteria to test convergence.

2 De Morgan and Bertrand's Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms ($a_n > 0$).

Define the limit L as:

$$L = \lim_{n\to\infty} \left[\left(n \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right) \ln(n) \right]$$

Convergence Rules

- If $L > 1$, the series $\sum a_n$ **converges**.
- If $L < 1$, the series $\sum a_n$ **diverges**.
- If $L = 1$, the test is **inconclusive**.

3 Bertrand's Test Extension

If $L = 1$, Bertrand's second-order logarithmic test is evaluated:

$$B = \lim_{n\to\infty} \left[\left(\left(n \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right) \ln(n) - 1 \right) \ln(\ln n) \right]$$

- If $B > 1$, the series **converges**.
- If $B < 1$, the series **diverges**.

4 Worked Example

Test the series:

$$\sum_{n=1}^{\infty} \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2}$$

Step 1: Ratio of Consecutive Terms

$$\frac{a_n}{a_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{4n^2+8n+4}{4n^2+4n+1}$$

Step 2: Raabe's Test Evaluation

$$R = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{4n^2+3n}{4n^2+4n+1} = 1 \quad \text{(Fails)}$$

Step 3: De Morgan and Bertrand's Test Evaluation

Subtract 1 from the Raabe expression:

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1 = \frac{4n^2+3n}{4n^2+4n+1} - 1 = \frac{-n-1}{4n^2+4n+1}$$

Multiply by $\ln n$ and take the limit:

$$L = \lim_{n\to\infty} \left(\frac{-n-1}{4n^2+4n+1} \ln n\right) = \lim_{n\to\infty} \left(-\frac{\ln n}{4n}\right) = 0$$

Conclusion

Since $L = 0 < 1$, the series **diverges**.