Dimension of Subspaces and Quotient Spaces

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1 Dimension of a Vector Space

Let V be a vector space over a field $\mathbb{F}$.

Definition 1.1. *A set*

$$\{v_1, v_2, \ldots, v_n\}$$

is called a basis of V if it is linearly independent and spans V.

The number of vectors in a basis of V is called the ***dimension*** *of V, denoted by $\dim V$.*

Thus, if

$$\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$$

is a basis of V, then

$$\dim V = n.$$

Example 1.1. *Consider*

$$V = \mathbb{R}^3.$$

The standard basis is

$$\mathcal{B} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

Therefore,

$$\boxed{\dim \mathbb{R}^3 = 3}.$$

2 Dimension of a Subspace

Let W be a subspace of a finite-dimensional vector space V.

Since W is itself a vector space, it has a basis. If

$$\mathcal{B}_W = \{w_1, w_2, \ldots, w_k\}$$

is a basis of W, then

$$\boxed{\dim W = k}.$$

Moreover,

$$\boxed{\dim W \leq \dim V}.$$

Example 2.1. *Let*

$$W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}.$$

From

$$x + y + z = 0,$$

we get

$$z = -x - y.$$

Hence,

$$(x, y, z) = (x, y, -x - y).$$

Therefore,

$$(x, y, z) = x(1, 0, -1) + y(0, 1, -1).$$

Thus,

$$W = \operatorname{span}\{(1, 0, -1), (0, 1, -1)\}.$$

The two vectors are linearly independent. Hence,

$$\boxed{\dim W = 2}.$$

3 Extension of a Basis

Theorem 3.1. *Let V be a finite-dimensional vector space and let W be a subspace of V. Any basis of W can be extended to a basis of V.*

Proof. Suppose

$$\{w_1, w_2, \dots, w_k\}$$

is a basis of W.

If this set already spans V, then it is a basis of V.

Otherwise, choose

$$v_{k+1} \in V$$

such that

$$v_{k+1} \notin W.$$

Then

$$\{w_1, \dots, w_k, v_{k+1}\}$$

is linearly independent.

Continuing this process, we obtain a basis of V of the form

$$\{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}.$$

Therefore,

$$\dim W = k, \qquad \dim V = n,$$

and hence

$$\boxed{\dim W \leq \dim V}.$$

□

4 Sum of Two Subspaces

Let W_1 and W_2 be subspaces of V.

The sum of W_1 and W_2 is defined by

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1,\ w_2 \in W_2\}.$$

Theorem 4.1 (Grassmann Dimension Formula)**.** *For finite-dimensional subspaces W_1 and W_2,*

$$\boxed{\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).}$$

Example 4.1. *Let*

$$W_1 = \operatorname{span}\{(1,0,0),(0,1,0)\}$$

and

$$W_2 = \operatorname{span}\{(0,1,0),(0,0,1)\}.$$

Then

$$\dim W_1 = 2, \qquad \dim W_2 = 2.$$

Also,

$$W_1 \cap W_2 = \operatorname{span}\{(0,1,0)\},$$

so

$$\dim(W_1 \cap W_2) = 1.$$

Hence,

$$\dim(W_1 + W_2) = 2 + 2 - 1 = 3.$$

Thus,

$$\boxed{W_1 + W_2 = \mathbb{R}^3}.$$

5 Direct Sum

Definition 5.1. *Two subspaces W_1 and W_2 of V are said to form a* ***direct sum*** *if*

$$W_1 \cap W_2 = \{0\}.$$

In this case we write

$$V = W_1 \oplus W_2.$$

For a direct sum,

$$\boxed{\dim(W_1 \oplus W_2) = \dim W_1 + \dim W_2.}$$

Example 5.1. *Let*

$$W_1 = \operatorname{span}\{(1,0,0)\}$$

and

$$W_2 = \operatorname{span}\{(0,1,0)\}.$$

Since

$$W_1 \cap W_2 = \{(0,0,0)\},$$

we have

$$W_1 \oplus W_2 = \operatorname{span}\{(1,0,0),(0,1,0)\}.$$

Therefore,

$$\boxed{\dim(W_1 \oplus W_2) = 2}.$$

6 Quotient Spaces

Let V be a vector space and W a subspace of V.

Definition 6.1. *The **quotient space** of V by W, denoted by*

$$V/W,$$

is the set of all cosets of W in V.

For $v \in V$, the coset of v is

$$v + W = \{v + w : w \in W\}.$$

Hence,

$$V/W = \{v + W : v \in V\}.$$

7 Equality of Cosets

Two cosets $v_1 + W$ and $v_2 + W$ are equal if and only if

$$\boxed{v_1 - v_2 \in W.}$$

Equivalently,

$$\boxed{v_1 + W = v_2 + W \iff v_1 - v_2 \in W.}$$

Example 7.1. *Let*

$$V = \mathbb{R}^2$$

and

$$W = \text{span}\{(1, 0)\}.$$

Then

$$W = \{(x, 0) : x \in \mathbb{R}\}.$$

Consider

$$(2, 3) + W.$$

We have

$$(2, 3) + W = \{(2, 3) + (x, 0) : x \in \mathbb{R}\}.$$

Therefore,

$$(2, 3) + W = \{(2 + x, 3) : x \in \mathbb{R}\}.$$

Thus,

$$\boxed{(2, 3) + W = \{(x, 3) : x \in \mathbb{R}\}.}$$

8 Vector Space Operations on V/W

For

$$v_1 + W, \quad v_2 + W \in V/W,$$

define addition by

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W.$$

For $\alpha \in \mathbb{F}$, define scalar multiplication by

$$\alpha(v + W) = \alpha v + W.$$

These operations are well-defined, and consequently

$$\boxed{V/W \text{ is a vector space.}}$$

9 Dimension of a Quotient Space

Theorem 9.1. *Let V be a finite-dimensional vector space and let W be a subspace of V. Then*

$$\boxed{\dim(V/W) = \dim V - \dim W.}$$

Proof. Let

$$\{w_1, w_2, \ldots, w_k\}$$

be a basis of W.

Extend this basis to a basis of V:

$$\{w_1, \ldots, w_k, v_{k+1}, \ldots, v_n\}.$$

Then

$$\dim W = k, \qquad \dim V = n.$$

The set

$$\{v_{k+1} + W, \ldots, v_n + W\}$$

forms a basis of V/W.

Hence,

$$\dim(V/W) = n - k.$$

Therefore,

$$\boxed{\dim(V/W) = \dim V - \dim W.}$$

□

Example 9.1. *Let*

$$V = \mathbb{R}^3$$

and

$$W = \operatorname{span}\{(1, 0, 0), (0, 1, 0)\}.$$

Then

$$\dim V = 3, \qquad \dim W = 2.$$

Therefore,

$$\boxed{\dim(V/W) = 3 - 2 = 1.}$$

A basis of V/W is

$$\boxed{\{(0, 0, 1) + W\}.}$$

Example 9.2. *Let*

$$V = \mathbb{R}^4$$

and

$$W = \operatorname{span}\{(1,0,0,0),(0,1,0,0)\}.$$

Then

$$\dim V = 4, \qquad \dim W = 2.$$

Thus,

$$\boxed{\dim(V/W) = 4 - 2 = 2.}$$

A basis of V/W is

$$\boxed{\{(0,0,1,0)+W,(0,0,0,1)+W\}.}$$

10 Quotient Spaces of Polynomial Spaces

Consider

$$V = P_3(\mathbb{R}),$$

the vector space of all real polynomials of degree at most 3.

A basis is

$$\{1, x, x^2, x^3\}.$$

Therefore,

$$\dim V = 4.$$

Let

$$W = \operatorname{span}\{1, x\}.$$

Then

$$\dim W = 2.$$

Hence,

$$\boxed{\dim(V/W) = 4 - 2 = 2.}$$

A basis of V/W is

$$\boxed{\{x^2 + W,\ x^3 + W\}.}$$

11 Quotient Spaces and Linear Transformations

Let

$$T : V \to U$$

be a linear transformation.

The kernel of T is

$$\ker T = \{v \in V : T(v) = 0\}.$$

The First Isomorphism Theorem states that

$$\boxed{V/\ker T \cong \operatorname{Im} T.}$$

Consequently,

$$\boxed{\dim(V/\ker T) = \dim(\operatorname{Im} T).}$$

Since

$$\dim(V/\ker T) = \dim V - \dim(\ker T),$$

we obtain the Rank–Nullity Theorem:

$$\boxed{\dim V = \dim(\ker T) + \dim(\operatorname{Im} T).}$$

Example 11.1. *Define*

$$T : \mathbb{R}^3 \to \mathbb{R}^2$$

by

$$T(x, y, z) = (x + y, y + z).$$

To find the kernel, solve

$$x + y = 0, \qquad y + z = 0.$$

Thus,

$$x = -y, \qquad z = -y.$$

Therefore,

$$(x, y, z) = y(-1, 1, -1).$$

Hence,

$$\ker T = \operatorname{span}\{(-1, 1, -1)\}.$$

Thus,

$$\dim(\ker T) = 1.$$

Since

$$\dim \mathbb{R}^3 = 3,$$

the Rank–Nullity Theorem gives

$$\dim(\operatorname{Im} T) = 3 - 1 = 2.$$

Therefore,

$$\boxed{\dim(\mathbb{R}^3/\ker T) = 2.}$$

Moreover,

$$\boxed{\mathbb{R}^3/\ker T \cong \operatorname{Im} T.}$$

12 Important Results

For a finite-dimensional vector space V and subspaces W, W_1, W_2,

$$\boxed{\dim W \le \dim V}$$

$$\boxed{\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)}$$

$$\boxed{\dim(W_1 \oplus W_2) = \dim W_1 + \dim W_2}$$

$$\boxed{\dim(V/W) = \dim V - \dim W}$$

and for a linear transformation $T : V \to U$,

$$\boxed{V/\ker T \cong \operatorname{Im} T}$$

and

$$\boxed{\dim V = \dim(\ker T) + \dim(\operatorname{Im} T).}$$