Q.3 Write value of $\iint_S \left(\nabla \times \vec{F}\right).d\vec{S}$ in line integral.

Q.4 Write value of line integral of $\vec{F} = 2x\hat{i} + x^2\hat{j}$ along x-axis from $x = 1$ to $x = 2$.

Q.5 If C is closed curve then write value of $\oint_C \vec{F}.d\,\vec{r}$ where $\vec{F}$ is conservative field.

Section – B (Short Answer type Question)

Q.6 Find the work done in the force field $\vec{F} = e^{y+2z}\left(\hat{i} + x\hat{j} + 2x\hat{k}\right)$ in moving the particle from $(0, 0, 0)$ to $(2, 2, 1)$.

Q.7 Evaluate $\int_C \left(xydx + xy^2dy\right)$ where C is boundary of square in xy-plane with vertices $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$ using Stoke's theorem.

Q.8 Evaluate $\int_C \vec{F}.d\,\vec{r}$ where $\vec{F} = -y\hat{i} + x\hat{j}$ & C is boundary of ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $z = 0$.

Q.9 Evaluate $\int_S \vec{F}.\hat{n}dS$ where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ & S is region bounded by $x^2 + y^2 = 4$ & plane $z = 0$ to $z = 3$.

Q.10 Evaluate $\iint_S \vec{F}.\hat{n}dS$ where $\vec{F} = \dfrac{\vec{r}}{r^3}$ & S is surface $x^2 + y^2 + z^2 = a^2$.

Section – C (Long Answer type Question)

Q.11 Verify Gauss divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ over the surface of cube bounded by $n = 0$, $y = 0$, $z = 0$, $x = 1$, $y = 1$, $z = 1$.

Q.12 Verify Gauss divergence theorem for $\vec{F} = \left(2x^2 - 3z\right)\hat{i} - 2xy\hat{j} - 4x\hat{k}$ over the region bounded by co-ordinate planes & $2x + 2y + z = 4$.

Q.13 Verify Gauss divergence theorem for $\vec{F} = x^2\,\hat{i} + y^2\hat{j} + z^2\hat{k}$ over the surface of ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Q.14 Verify Stoke's theorem for vector point function $\vec{F} = x^2\hat{i} + yx\hat{j}$ round the square in plane $z = 0$ whose sides are along straight lines $x = y = 0$ & $x = y = a$.

Q.15 Verify Stoke's theorem for $\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$ over the surface of sphere $x^2 + y^2 + z^2 = a^2$ above xy-plane.

3.16 Answers to Exercise

Ans.3 : $\int_C \vec{F}.d\,\vec{r}$	**Ans.4 :** 3
Ans.5 : 0	**Ans.6 :** $2e^4$
Ans.7 : $\frac{4}{3}$	**Ans.8 :** 0
Ans.9 : 84π	**Ans.10 :** 4π

References and Suggested Readings

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UNIT- 4

Tensor Analysis

Structure of the unit

4.0 Objectives

In this unit we are going to discuss about tensors and its properties.

After going through this unit you will be able to learn

- N-dimensional space
- Contravariant and covariant vectors

- Algebraic operations with tensors
- Contraction ,Direct product, Quotient rule
- Symmetric and anti-symmetric tensor, Pseudo-tensor

4.1 Introduction

The fundamental postulate of Physics is that the laws of nature are covariant. ***The meaning of covariant is that they have the same form in all reference frames***. Tensor formulation is a mathematical tool in which all the physical laws can be formulated in a covariant way.

The tensor formulation was originally given by G. Ricci and it became popular when Albert Einstein used it as a natural tool for the description of his general theory of relativity. It has became an important mathematical tool in almost every branch of theoretical physics such as Mechanics, Electrodynamics, Elasticity, Fluid mechanics etc. Tensor analysis is the generalization of vector calculus.

4.2 N-dimensional space

In three dimensional space (Cartesian system), the coordinates of a point are given by (x, y, z) where x, y, z are numbers. For the generalization of concept of space, from three dimensions to N-dimensions this representation is not suitable. An ordered set of N real variables $x^1, x^2, \ldots\ldots x^N$ can be associated with a point in space and will be called the coordinates of the point. All the points corresponding to all of the coordinates are said to form an N-dimensional space, denoted byV_N.

Transformation of Coordinates:-

The process of obtaining one set of numbers from the other is known as coordinate transformation. Consider two different N-dimensional spaces. Let us consider two sets of variables $(x^1, x^2, \ldots . x^N)$ and $(x'^1, x'^2, \ldots . x'^N)$. A transformation from $(x^1, x^2, \ldots . x^N)$ to the new set of variables $(x'^1, x'^2, \ldots . x'^N)$ through the equations

$$x'^i = f^i(x^1, x^2, \ldots . x^N) \qquad (1)$$

gives a transformation of coordinates. Here f^i is assumed to be single valued real function of the coordinates and possess continuous partial derivatives. This ensures the existence of inverse transformation and is given as

$$x^i = g^i(x'^1, x'^2, \ldots . x'^N) \qquad (2)$$

The suffixes or indices i, j in A^i_j are called superscript and subscript respectively.

Concepts of Scalar, Vector and Tensor: -

Scalar: A physical quantity that can be completely described by a real number. Example:- Temperature, mass, density, potential etc. The expression of its component is independent of the choice of the coordinate system.

Vector: A physical quantity that has both direction and length. Example:- Displacement, velocity, force, heat flow etc. The expression of its component is dependent of the choice of the coordinate system.

Tensor: A tensor defines an operation that transforms a vector to another vector. A tensor contains the information about the directions and the magnitudes in those directions.

4.3 Contravariant and Covariant Vectors

Contravariant vectors:-

A set of N functions $A^i(i = 1 \ldots N)$ of the N coordinates $x^i(i = 1 \ldots N)$ are said to be components of a contravariant vector if they transform according to the equation

$$\boxed{A'^i = \frac{\partial x'^i}{\partial x^j} A^j} \qquad (3)$$

on change of the coordinates x^i to x'^i.

Covariant vectors:-

A set of N functions $A_i(i = 1 \ldots N)$ of the N coordinates $x^i(i = 1 \ldots N)$ are said to be components of a covariant vector if they transform according to the equation

$$\boxed{A'_i = \frac{\partial x^j}{\partial x'^i} A_j} \qquad (4)$$

on change of the coordinates x^i to x'^i.

Only in Cartesian coordinate

$$\frac{\partial x'^i}{\partial x^j} = \frac{\partial x^j}{\partial x'^i} \qquad \dots (5)$$

So that ***there is no difference between covariant and contravariant transformations in Cartesian coordinates***. In other coordinate systems, eqn. (5) in general doesn't apply and there is difference between covariant and contravariant transformations in other coordinate systems. This is important in the curved Riemannian space of general relativity.

Definitions of tensors of rank 2:-

The rank goes as the number of partial derivatives in the definition:

0 for scalar, 1 for vector, 2 for a second rank tensor and so on.

A covariant tensor A_{ij} of rank two is transformed as

$$\boxed{A'_{ij} = \sum_{kl} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} A_{kl}} \qquad (6)$$

A contravariant tensor A^{ij} of rank two is transformed as

$$\boxed{A'^{ij} = \sum_{kl} \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} A^{kl}} \qquad (7)$$

A mixed tensor is contravariant in some indices and covariant in the others. A mixed tensor A^i_j of rank two is transformed as

$$\boxed{A'^i_j = \sum_{kl} \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} A^k_l} \qquad (8)$$

The components of a vector transform according to eqns. (6), (7), (8) yield entities that are independent of the choice of reference frame. That's why tensor analysis is important in physics.

Example 4.1 A covariant tensor has components $xy\ , 2y - z^2, xz$ in rectangular co-ordinates. Find its covariant components in spherical co-ordinates.

Sol. Let A_j denote the covariant component in rectangular co-ordinates

$$x^1 = x, x^2 = y, x^3 = z.$$

Then $$A_1 = xy = x^1x^2$$

$$A_2 = 2y - z^2 = 2x^2 - (x^3)^2$$

$$A_3 = xz = x^1x^3$$

Let A'_k denote the covariant component in spherical co-ordinates

$$x'^1 = r, x'^2 = \theta, x'^3 = \varphi.$$

Then $$A'_k = \frac{\partial x^j}{\partial x'^k} A_j \qquad (1)$$

In spherical coordinates

$$x = r\sin\theta\cos\varphi$$

$$or \quad x^1 = x'^1 \sin x'^2 \cos x'^3$$

$$y = r\sin\theta\sin\varphi$$

$$or \quad x^2 = x'^1 \sin x'^2 \sin x'^3$$

$$z = r\cos\theta$$

$$or \quad x^3 = x'^1 \cos x'^2$$

Therefore equation (1) yields the covariant component.

$$A'_1 = \frac{\partial x^1}{\partial x'^1} A_1 + \frac{\partial x^2}{\partial x'^1} A_2 + \frac{\partial x^3}{\partial x'^1} A_3$$

$$= (\sin x'^2 \cos x'^3)(x^1x^2) + (\sin x'^2 \sin x'^3)(2x^2 - (x^3)^2) + \cos x'^2 (x^1x^3)$$

$$= (\sin\theta\cos\varphi)(r^2\sin^2\theta\sin\varphi\cos\varphi) + (\sin\theta\sin\varphi)(2r\sin\theta\sin\varphi - r^2\cos^2\theta) + (\cos\theta)(r^2\sin\theta\cos\theta\cos\varphi)$$

$$A'_2 = \frac{\partial x^1}{\partial x'^2} A_1 + \frac{\partial x^2}{\partial x'^2} A_2 + \frac{\partial x^3}{\partial x'^2} A_3$$

$$= (r\cos\theta\cos\varphi)(r^2\sin^2\theta\sin\varphi\cos\varphi) + (r\cos\theta\sin\varphi)\,(2r\sin\theta\sin\varphi - r^2\cos^2\theta) + (-r\sin\theta)(r^2\sin\theta\cos\theta\cos\varphi)$$

$$A'_3 = \frac{\partial x^1}{\partial x'^3} A_1 + \frac{\partial x^2}{\partial x'^3} A_2 + \frac{\partial x^3}{\partial x'^3} A_3$$

$$= (-r\sin\theta\sin\varphi)(r^2\sin^2\theta\sin\varphi\cos\varphi) + (r\sin\theta\cos\varphi)\,(2r\sin\theta\sin\varphi - r^2\cos^2\theta) + 0.$$

4.4 Self Learning Exercise-I

Q.1 What is the rank of tensor A_{ij}^{k} ?

Q.2 Write the transformation of covariant tensor B_{ij}?

Q.3 Write one example of a mixed tensor of rank 3.

4.5 Algebraic Operations with Tensors

Kronecker Delta: -

The Kronecker delta is defined as

$$\boxed{\begin{array}{l} \delta_j^i = 1 \text{ If i } = \text{ j} \\ \delta_j^i = 0 \text{ If i} \neq \text{j} \end{array}}$$

So by definition of Kronecker delta we can write

$$\delta_1^1 = \delta_2^2 = \delta_3^3 = \cdots = \delta_N^N = 1$$
$$\delta_2^1 = \delta_3^2 = \cdots = 0$$

The coordinates $x^1, x^2, \dots, x^N$ are independent so Kronecker delta can also be written as

$$\boxed{\frac{\partial x^i}{\partial x^j} = \delta_j^i}$$

Similarly we can write

$$\frac{\partial x'^i}{\partial x'^j} = \delta_j^i$$

Addition and Subtraction of Tensors:-

The sum of two or more tensors of the same rank and type (i.e. same number of contra-variant and same numbers of covariant indices) is also a tensor of the same rank and type. Thus the sum of two tensors

$$C_j^i = A_j^i + B_j^i$$

is also a tensor.

We can also subtract two tensors provided they are of the same rank and type. The difference of two tensors of same rank and type is another tensor of the same rank and type.

$$C_j^i = A_j^i - B_j^i$$

Theorem: - The sum or difference of two tensors of the same rank and type is again a tensor of the same rank and type.

Proof: - Let A_j^i and B_j^i are two tensors in coordinate system x^i and having the following transformation relations in the coordinate system x'^i

$$A'^p_q = A_j^i \frac{\partial x'^p}{\partial x^i} \frac{\partial x^j}{\partial x'^q} \qquad (1)$$

$$B'^p_q = B_j^i \frac{\partial x'^p}{\partial x^i} \frac{\partial x^j}{\partial x'^q} \qquad (2)$$

From (1) and (2) we get

$$(A'^p_q \pm B'^p_q) = (A_j^i \pm B_j^i) \frac{\partial x'^p}{\partial x^i} \frac{\partial x^j}{\partial x'^q}$$

Which shows that $(A \pm B)$ follows the same law of transformation as in A and B. Hence $(A \pm B)$ is also a tensor of the same rank and type.

4.6 Contraction

If in a tensor we put one contra-variant and one covariant indices equal (i.e. same) then the summation over equal indices is to be taken according to the summation convention. The process is called contraction of a tensor.

Consider a tensor A_{pqr}^{ij} of rank five. If we put$j = i$, we get A_{pqr}^{ij} and is a tensor of rank 3 obtained by contracting A_{pqr}^{ij}.

Theorem :- If in a mixed tensor, contra variant of rank p and covariant of rank q, we equate a covariant and contra variant index and sum with regard to that index then the resulting set of N^{p+q-2} sums is mixed tensor, contra variant of rank (p-1) and covariant of rank (q-1).

Proof: Consider a mixed tensor A_{lmn}^{ij} of rank five, contra variant of rank two and covariant of rank three.

$$A'^{ij}_{lmn} = A^{st}_{uvw} \frac{\partial x'^i}{\partial x^s} \frac{\partial x'^j}{\partial x^t} \frac{\partial x^u}{\partial x'^l} \frac{\partial x^v}{\partial x'^m} \frac{\partial x^w}{\partial x'^n}$$

Let $j = n$, we get

$$A'^{in}_{lmn} = A^{st}_{uvw} \frac{\partial x'^i}{\partial x^s} \frac{\partial x'^n}{\partial x^t} \frac{\partial x^u}{\partial x'^l} \frac{\partial x^v}{\partial x'^m} \frac{\partial x^w}{\partial x'^n}$$

$$= A^{st}_{uvw} \frac{\partial x'^i}{\partial x^s} \frac{\partial x^u}{\partial x'^l} \frac{\partial x^v}{\partial x'^m} \left(\frac{\partial x^w}{\partial x'^n} \frac{\partial x'^n}{\partial x^t} \right)$$

$$= A^{st}_{uvw} \frac{\partial x'^i}{\partial x^s} \frac{\partial x^u}{\partial x'^l} \frac{\partial x^v}{\partial x'^m} \frac{\partial x^w}{\partial x^t}$$

$$= A^{st}_{uvw} \frac{\partial x'^i}{\partial x^s} \frac{\partial x^u}{\partial x'^l} \frac{\partial x^v}{\partial x'^m} \delta^w_t$$

$$= A^{kw}_{uvw} \frac{\partial x'^i}{\partial x^k} \frac{\partial x^u}{\partial x'^l} \frac{\partial x^v}{\partial x'^m}$$

$$= A^{k}_{uv} \frac{\partial x'^i}{\partial x^k} \frac{\partial x^u}{\partial x'^l} \frac{\partial x^v}{\partial x'^m}$$

In this last expression we have put $A^{kw}_{uvw} = A^k_{uv}$. This is the law of transformation of a tensor of rank three.

Thus $A'^{in}_{lmn} = A'^{i}_{lm}$ is a tensor of rank 3 contra variant of rank 2-1 i.e. 1 and covariant of rank 3-1 i.e. 2.

4.7 Direct Product

Let A_i is a covariant vector (first rank tensor) and B^j is a contra-variant vector (first rank tensor) then component of A_i and B^j may be multiplied component by component to give the general term $A_i B^j$.

$$A_i B^j = \frac{\partial x^k}{\partial x'^i} A_k \frac{\partial x'^j}{\partial x^l} B^l$$

$$= \frac{\partial x^k}{\partial x'^i} \frac{\partial x'^j}{\partial x^l} \left(A_k B^l \right)$$

Contracting, we get

$$A'_i B'^i = A_k B^k$$

The operation of adjoining two vectors A_i and B^j is known as the direct product of tensors. The product of two vectors is a tensor of rank two. In general, the direct product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors.

$$A_j^i B^{kl} = C_j^{ikl}$$

Where C_j^{ikl} is a tensor of fourth rank. The direct product is a technique for creating new higher-rank tensors.

The word “tensor product” refers to a way of constructing a big vector space out of two (or more) smaller vector spaces. If a vector V is n-dimensional and a vector is m-dimensional then the product of these two vector spaces is nm-dimensional.

In Quantum mechanics, for each dynamical degree of freedom we associate a Hilbert space. For example, a free particle in has three dynamical degrees of freedom p_x, p_y, p_z in three dimensional system. Note that we can specify only p_x or x, but not both and hence each dimension gives only one degree of freedom but in classical mechanics you have two

Example 4.2 Prove that $A_i B^i$ is invariant if A_i is covariant tensor and B^i is contravariant tensor.

Sol. By the law of transformation of tensors we have

$$A'_i = \frac{\partial x^k}{\partial x'^i} A_k$$

$$B'^i = \frac{\partial x'^i}{\partial x^l} B^l$$

Then we can get

$$A'_i B'^i = A_k B^l \frac{\partial x^k}{\partial x'^i} \frac{\partial x'^i}{\partial x^l}$$

$$= A_k B^l \delta_l^k$$

$$= A_k (\delta_l^k B^l)$$

$$A'_i B'^i = A_k B^k = A_i B^i$$

Hence $A_i B^i$ is an invariant.

Theorem: - The product of two tensors is a tensor of whose rank (or order) is the sum of the ranks of the two tensors.

Proof: Consider two tensors A_k^{ij} and B_q^p. Let the product of these two tensors be a tensor C_{kq}^{ijp}, that is

$$C_{kq}^{ijp} = A_k^{ij} B_q^p \qquad (1)$$

Now we have to show that C_{kq}^{ijp} is a tensor of rank 5. We know that

$$A_w'^{uv} = A_k^{ij} \frac{\partial x'^u}{\partial x^i} \frac{\partial x'^v}{\partial x^j} \frac{\partial x^k}{\partial x'^w} \qquad (2)$$

$$B_s'^r = B_q^p \frac{\partial x'^r}{\partial x^p} \frac{\partial x^q}{\partial x'^s} \qquad (3\ )$$

Multiplying equation (2) and (3), we get

$$A_w'^{uv} B_s'^r = A_k^{ij} B_q^p \frac{\partial x'^u}{\partial x^i} \frac{\partial x'^v}{\partial x^j} \frac{\partial x^k}{\partial x'^w} \frac{\partial x'^r}{\partial x^p} \frac{\partial x^q}{\partial x'^s}$$

Using (1)

$$= C_{kq}^{ijp} \frac{\partial x'^u}{\partial x^i} \frac{\partial x'^v}{\partial x^j} \frac{\partial x^k}{\partial x'^w} \frac{\partial x'^r}{\partial x^p} \frac{\partial x^q}{\partial x'^s}$$

We may write above equation as

$$C_{ws}'^{uvr} = C_{kq}^{ijp} \frac{\partial x'^u}{\partial x^i} \frac{\partial x'^v}{\partial x^j} \frac{\partial x^k}{\partial x'^w} \frac{\partial x'^r}{\partial x^p} \frac{\partial x^q}{\partial x'^s} \qquad (4)$$

Relation (4) is the law of transformation of a mixed tensor of rank five. Hence C_{kq}^{ijp} is a mixed tensor of rank 5.

Hence the theorem is proved.

Example 4.3 If A^μ and B_υ are compononents of a contravariant and covariant tensor of rank one, the prove that

$C_\upsilon^\mu = A^\mu B_\upsilon$ are the components of a mixed tensor of rank two.

Sol. Using tensor transformation

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} A^\alpha$$

$$B'_{\upsilon} = \frac{\partial x^{\beta}}{\partial x'^{\upsilon}} B_{\beta}$$

Thus $C'^{\mu}_{\upsilon} = A'^{\mu} B'_{\upsilon} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} A^{\alpha} \frac{\partial x^{\beta}}{\partial x'^{\upsilon}} B_{\beta}$

$$= \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\upsilon}} A^{\alpha} B_{\beta}$$

$$\Rightarrow C'^{\mu}_{\upsilon} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\upsilon}} C^{\alpha}_{\beta}$$

Hence above equation is transformation equation for a mixed tensor of rank two.

4.8 Quotient rule

By this law we can test whether a given quantity is a tensor or not. Suppose we are given a quantity X and we don't know whether X is a tensor or not. To test X, we take product of X with an arbitrary tensor, if this product is tensor then X is also a tensor. This is called Quotient law. The Quotient law is a simple indirect test which can be used to ascertain whether a set of quantities form the components of a tensor.

Theorem: - If the product of a set of quantities A^{ijk} with an arbitrary tensor B^{p}_{ij} yield a non-zero tensor C^{pk} then the quantities A^{ijk} are the components of a tensor.

Proof: We consider an arbitrary co-ordinate transformation $x^{i} \to x'^{i}$, from this transformation $A^{ijk} \to A'^{ijk}, B^{p}_{ij} \to B'^{p}_{ij}, C^{pk} \to C'^{pk}$.

Consider the product

$$C^{pk} = A^{ijk} B^{p}_{ij} \qquad (1)$$

In transformed coordinates x'^{i}, equation (1) becomes

$$C'^{pk} = A'^{ijk} B'^{p}_{ij} \qquad (2)$$

But we have

$$C'^{pk} = C^{qr} \frac{\partial x'^{p}}{\partial x^{q}} \frac{\partial x'^{k}}{\partial x^{r}} \qquad (3)$$

$$B'^{p}_{ij} = B^{l}_{mn} \frac{\partial x'^{p}}{\partial x^{l}} \frac{\partial x^{m}}{\partial x'^{i}} \frac{\partial x^{n}}{\partial x'^{j}} \qquad (4)$$

Substituting equation (3) and (4) into (2), we get

$$C^{qr}\frac{\partial x'^{p}}{\partial x^{q}}\frac{\partial x'^{k}}{\partial x^{r}} = A'^{ijk}\,B^{l}_{mn}\frac{\partial x'^{p}}{\partial x^{l}}\frac{\partial x^{m}}{\partial x'^{i}}\frac{\partial x^{n}}{\partial x'^{j}}$$

$$A^{ijk}B^{q}_{ij}\frac{\partial x'^{p}}{\partial x^{q}}\frac{\partial x'^{k}}{\partial x^{r}} = A'^{ijk}\,B^{l}_{mn}\frac{\partial x'^{p}}{\partial x^{l}}\frac{\partial x^{m}}{\partial x'^{i}}\frac{\partial x^{n}}{\partial x'^{j}}$$

By changing dummy indices $q \to l; i \to m$ and $j \to n$, we get

$$A^{mnr}B^{l}_{mn}\frac{\partial x'^{p}}{\partial x^{l}}\frac{\partial x'^{k}}{\partial x^{r}} = A'^{ijk}\,B^{l}_{mn}\frac{\partial x'^{p}}{\partial x^{l}}\frac{\partial x^{m}}{\partial x'^{i}}\frac{\partial x^{n}}{\partial x'^{j}}$$

$$\frac{\partial x'^{p}}{\partial x^{l}}\left[A'^{ijk}\frac{\partial x^{m}}{\partial x'^{i}}\frac{\partial x^{n}}{\partial x'^{j}} - A^{mnr}\frac{\partial x'^{k}}{\partial x^{r}}\right]B^{l}_{mn} = 0 \qquad (5)$$

We know that

$$\frac{\partial x^{s}}{\partial x'^{p}}\left[\frac{\partial x'^{p}}{\partial x^{l}}B^{l}_{mn}\right] = \frac{\partial x^{s}}{\partial x^{l}}B^{l}_{mn} = B^{s}_{mn} \qquad (6)$$

Multiplying equation (5) by $\frac{\partial x^{s}}{\partial x'^{p}}$ and substituting equation (6), one gets

$$\left[A'^{ijk}\frac{\partial x^{m}}{\partial x'^{i}}\frac{\partial x^{n}}{\partial x'^{j}} - A^{mnr}\frac{\partial x'^{k}}{\partial x^{r}}\right]B^{s}_{mn} = 0 \qquad (7)$$

Since equation is valid for arbitrary B^{s}_{mn}, therefore the quantity under bracket is zero i.e.

$$A'^{ijk}\frac{\partial x^{m}}{\partial x'^{i}}\frac{\partial x^{n}}{\partial x'^{j}} = A^{mnr}\frac{\partial x'^{k}}{\partial x^{r}}$$

Multiplying both sides by $\frac{\partial x'^{s}}{\partial x^{m}}\frac{\partial x'^{t}}{\partial x^{n}}$, one gets

$$A'^{ijk}\frac{\partial x'^{s}}{\partial x'^{i}}\frac{\partial x'^{t}}{\partial x'^{j}} = A^{mnr}\frac{\partial x'^{s}}{\partial x^{m}}\frac{\partial x'^{t}}{\partial x^{n}}\frac{\partial x'^{k}}{\partial x^{r}}$$

L.H.S. is non zero for $i = s$ and $j = t$ and hence

$$A'^{ijk} = A^{mnr}\frac{\partial x'^{i}}{\partial x^{m}}\frac{\partial x'^{j}}{\partial x^{n}}\frac{\partial x'^{k}}{\partial x^{r}} \qquad (8)$$

From equation (8), we see that A^{mnr} is a tensor of third rank and contra-variant in all indices. This completes the proof of Quotient law.

4.9 Symmetric and Anti-Symmetric Tensor

Based on permutation of the indices a tensor can be of two types:-

1. Symmetric tensor
2. Anti-symmetric tensor

A tensor is said to be symmetric in two indices of the same type i.e. both covariant or both contravariant, if the value of any component is not changed by permuting them. If for a tensor A_{ij}

$$\boxed{A_{ij} = A_{ji}}$$

Then it is called ***symmetric tensor***.

A tensor is said to be anti-symmetric in two indices of the same type i.e. both covariant or both contravariant, if the value of any component changes its sign by permuting them. If for a tensor A_{ij}

$$\boxed{A_{ij} = -A_{ji}}$$

Then it is called ***anti-symmetric*** tensor. A general tensor can be split up into a symmetric and an anti-symmetric part:

$$\boxed{A_{ij} = \frac{1}{2}\left(A_{ij} + A_{ji}\right) + \frac{1}{2}\left(A_{ij} - A_{ji}\right)}$$

Where the first part in the right hand side is symmetric part and second part is anti-symmetric part.

Theorem:- A symmetric tensor of rank two has at most $\frac{1}{2}N(N+1)$ different components in N - dimensional vector space.

Proof: Let A_{ij} be a tensor of rank two. The number of its all components in N-dimensional vector space is N^2. All the components of A_{ij} are

$$\begin{matrix} A_{11} & A_{12} & A_{13} & A_{14} \dots & A_{1N} \\ A_{21} & A_{22} & A_{23} & A_{24} \dots & A_{2N} \\ A_{31} & A_{32} & A_{33} & A_{34} \dots & A_{3N} \\ \dots & \dots & \dots & \dots. \quad \dots. & \dots. \\ A_{N1} & A_{N2} & A_{N3} & A_{N4} \dots & A_{NN} \end{matrix}$$