Example (2) solve

$$x^2 \frac{dy}{dx} = 4x^2 + 7xy + 2y^2$$

Solution: Given differential equation can be written as

$$\frac{dy}{dx} = \frac{4x^2 + 7xy + 2y^2}{x^2} \quad -①$$

putting $y = vx$, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad -②$$

From eqns ① and ②, we have

$$v + x\frac{dv}{dx} = \frac{4x^2 + 7x(vx) + 2(vx)^2}{x^2}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = 4 + 7v + 2v^2$$

$$\Rightarrow \quad x\frac{dv}{dx} = 4 + 7v + 2v^2 - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = 4 + 6v + 2v^2$$

$$\Rightarrow \quad \frac{dv}{4 + 6v + 2v^2} = \frac{dx}{x}$$

$$\Rightarrow \quad \frac{dv}{2[2 + 3v + v^2]} = \frac{dx}{x}$$

$$\Rightarrow \quad \frac{dv}{2[v^2 + v + 2v + 2]} = \frac{dx}{x}$$

$$\Rightarrow \frac{du}{2[u(u+1)+2(u+1)]} = \frac{dx}{x}$$

$$\Rightarrow \frac{du}{2(u+1)(u+2)} = \frac{dx}{x}$$

$$\Rightarrow \frac{[(u+2)-(u+1)]\,du}{2(u+1)(u+2)} = \frac{dx}{x}$$

$$\Rightarrow \left[\frac{1}{u+1} - \frac{1}{u+2}\right] du = \frac{2\,dx}{x}$$

On integration, this gives

$$\ln(u+1) - \ln(u+2) = \ln x^2 + \ln c$$

$$\Rightarrow \ln\left(\frac{u+1}{u+2}\right) = \ln(cx^2) \quad \left[\text{where } c \text{ is an integration constant}\right]$$

$$\Rightarrow cx^2 = \frac{u+1}{u+2} \quad -③$$

put $u = \frac{y}{x}$ in eqn ③, we get

$$\boxed{cx^2 = \frac{y+x}{y+2x}} \quad \text{Ans}$$

Home work

Solve the following differential equation:

① $(x \operatorname{cosec}(\frac{y}{x}) - y)dx + x\,dy = 0$

② $x\,dy - y\,dx = \sqrt{x^2+y^2}\;dx$

③ $\left(x\cos(\frac{y}{x}) + y\sin\frac{y}{x}\right)y - \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x\frac{dy}{dx} = 0$

④ $y^2 dy = x(x\,dy - y\,dx)e^{x/y}$

(5) $(x\cos^2\frac{y}{x} - y)dx + x\,dy = 0$; when $x = 1$, $y = \frac{\pi}{4}$

(6) $y\tan^{-1}\left(\frac{x}{y}\right)dx + \left[y - x\tan^{-1}\left(\frac{x}{y}\right)\right]dy = 0$

(7) $\frac{dy}{dx} = \frac{y}{x} - \frac{x}{y}e^{-y/x}$

(8) $\frac{dy}{dx} = \frac{y\sin\left(\frac{x}{y}\right)}{y + x\sin\left(\frac{x}{y}\right)}$

Equations reducible to homogeneous form:

Let the given differential equation be of the form

$$\frac{dy}{dx} = g\left(\frac{a_1x + a_2y + a_3}{b_1x + b_2y + b_3}\right),\text{ where}$$

a_1, a_2, a_3, b_1, b_2 and b_3 are constants.

We put $x = \xi + h$, $y = \eta + K$, where h and K are constants to be chosen in such a way that

$$\frac{a_1x + a_2y + a_3}{b_1x + b_2y + b_3} = \frac{a_1\xi + a_2\eta}{b_1\xi + b_2\eta}$$

$$\Rightarrow \frac{a_1(\xi + h) + a_2(\eta + K) + a_3}{b_1(\xi + h) + b_2(\eta + K) + b_3} = \frac{a_1\xi + a_2\eta}{b_1\xi + b_2\eta}$$

$$\Rightarrow \frac{(a_1\xi + a_2\eta) + (a_1h + a_2K + a_3)}{\cancel{b_1(\xi + h)}(b_1\xi + b_2\eta) + (b_1h + b_2K + b_3)} = \frac{a_1\xi + a_2\eta}{b_1\xi + b_2\eta}$$

$$\Rightarrow \left.\begin{aligned} a_1h + a_2k + a_3 &= 0 \\ b_1h + b_2k + b_3 &= 0 \end{aligned}\right\}$$ Simultaneous equations in h and K, hence we can solve it for h and K.

Above equations can be solved uniquely for h and K

if $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0$ i.e; $a_1b_2 - a_2b_1 \neq 0$